LOWER BOUNDS OF COPSON TYPE FOR THE TRANSPOSE OF MATRICES ON WEIGHTED SEQUENCE SPACES

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ABSTRACT. Let \( A = (a_{n,k})_{n,k \geq 0} \) be a non-negative matrix. Denote by \( L_{w,p,q}(A) \), the supremum of those \( L \), satisfying the following inequality:

\[
\left( \sum_{n=0}^{\infty} w_n \left( \sum_{k=0}^{\infty} a_{n,k} x_k \right)^q \right)^{\frac{1}{q}} \geq L \left( \sum_{k=0}^{\infty} w_k x_k^p \right)^{\frac{1}{p}},
\]

where, \( x \geq 0 \) and \( x \in l_p(w) \) and also \( w = (w_n) \) is a decreasing, non-negative sequence of real numbers. If \( p = q \), then we use \( L_{w,p}(A) \) instead of \( L_{w,p,p}(A) \). Here, we focus on the evaluation of \( L_{w,p}(A^t) \) for a lower triangular matrix \( A \), where, \( 0 < p < 1 \). In particular, we apply our results to summability matrices, weighted mean matrices, Nörlund matrices. Our results also generalize some results in Chen and Wang [C.-P. Chen and K.-Z. Wang, J. Math. Anal. Appl. 341 (2008) 1284-1294], Foroutannia and Lashkaripour [D. Foroutannia and R. Lashkaripour, Lobachevskii J. Math. 27 (2007) 15-29], and Lashkaripour and Foroutannia [R. Lashkaripour and D. Foroutannia, J. Sci. Islam. Repub. Iran 18 (2007) 49-56].
1. Introduction

Let \( p \in \mathbb{R} \setminus \{0\} \) and let \( l^p(w) \) denote the space of all real sequences \( x = \{x_k\}_{k=0}^{\infty} \) such that \( \|x\|_{w,p} := \left( \sum_{k=0}^{\infty} w_k x_k^p \right)^{1/p} < \infty \), where, \( w = (w_n)_{n=0}^{\infty} \) is a decreasing, non-negative sequence of real numbers with \( \sum_{n=0}^{\infty} \frac{w_n}{n+1} = \infty \) with \( w_0 = 1 \).

We write \( x \geq 0 \) if \( x_k \geq 0 \), for all \( k \). We also write \( x \uparrow \) for the case that \( x_0 \leq x_1 \leq \cdots \leq x_n \leq \cdots \). The symbol \( x \downarrow \) is defined in a similar way. For \( p, q \in \mathbb{R} \setminus \{0\} \), the lower bound involved here is the number \( L_{w,p,q}(A) \), which is defined as the supremum of those \( L \) obeying the following inequality:

\[
\left( \sum_{n=0}^{\infty} w_n \left( \sum_{k=0}^{\infty} a_{n,k} x_k \right)^q \right)^{1/q} \geq L \left( \sum_{k=0}^{\infty} w_k x_k^p \right)^{1/p}, \quad (x \geq 0, x \in l^p(w)),
\]

where, \( A \geq 0 \), that is, \( A = (a_{n,k})_{n,k \geq 0} \) is a non-negative matrix. We have

\[
L_{w,p,q}(A) \leq \|A\|_{w,p,q}.
\]

In [3], the author obtained \( L_{w,p}(C(1)^t) = p, (0 < p < 1) \), where, \((.)^t\) denotes the transpose of \((.)\) and \( C(1) = (a_{n,k})_{n,k \geq 0} \) is the Cesaro matrix defined by

\[
a_{n,k} = \begin{cases} \frac{1}{n+1} & 0 \leq k \leq n \\ 0 & \text{otherwise}. \end{cases}
\]

This is an analogue of Copson’s result [2, Eq. (1.1)] (see also [4], Theorem 344) for weighted sequence space \( l^p(w) \) and has been generalized by Foroutannia [3]. He extended it in [3, Theorem 2.7.17 and Theorem 2.7.19] to those summability matrices \( A \), whose rows are increasing or decreasing. Also, he gave upper bounds or lower bounds for \( L_{w,p}(A) \), for such \( A \). For the case of Hausdorff matrices, the related result with \( 0 < p < 1 \) has been established in [3, Theorem 4.3.2], giving a Hardy-type formula for \( L_{w,p}(H_1^p) \).

Obviously, the lower bound problems of Copson type for the weighted mean matrices, \( (A^W_W)^M = (a_{n,k})_{n,k \geq 0} \), and the Nörlund matrices, \( (A^N_W)^M = (b_{n,k})_{n,k \geq 0} \), or more generally for the summability matrices on weighted sequence spaces are still less satisfactory (cf. [1, problem 4.20]), where
the weighted mean matrices and the Nörlund matrices are defined as:

\[ a_{n,k} = \begin{cases} \frac{w'_k}{W'_n} & 0 \leq k \leq n \\ 0 & \text{otherwise} \end{cases} \]

and

\[ b_{n,k} = \begin{cases} \frac{w'_{n-k}}{W'_n} & 0 \leq k \leq n \\ 0 & \text{otherwise} \end{cases} \]

Here, \( W'_n = \sum_{k=0}^{n} w'_k \), and \( w' = (w'_n) \) is a non-negative sequence with \( w'_0 > 0 \).

Here we are concerned with the problem of finding \( L_{w,p}(A^t) \) and \( L_{w,p^*}(A) \) (see Theorem 2.3), where, \( 0 < p < 1, \frac{1}{p} + \frac{1}{p^*} = 1 \) and \( A \) is a non-negative lower triangular matrix. Our result gives a lower estimate for these two values in terms of the constant \( M \), defined by:

(1.1) \[ a_{n,k} \leq Ma_{n,j}, \quad (0 \leq k \leq j \leq n). \]

Here, \( M \geq 1 \). We shall assume that \( M \) is the smallest value appearing in (1.1). If (1.1) is not satisfied, then we set \( M = \infty \). As a consequence, we prove that Theorem 2.3 generalizes some works of Lashkaripour and Foroutannia ([3], pp.53-54). Also, we obtain lower estimate and upper estimate for the weighted mean matrix and the Nörlund matrix in some cases.

2. Main Result

The purpose of this section is to establish general lower bounds for \( L_{w,p}(A^t) \) and \( L_{w,p^*}(A) \), where, \( 0 < p < 1, \frac{1}{p} + \frac{1}{p^*} = 1 \) and \( A \) is a non-negative lower triangular matrix. First, we generalize Lemma 5.2 of [2] to the weighted sequence space \( l_p(w) \).

**Lemma 2.1.** Suppose that \( 0 < p < 1, \frac{1}{p} + \frac{1}{p^*} = 1 \) and \( N \in \mathbb{N} \). Let \( C^1_N = (c_{n,k}(N))_{n,k \geq 0} \) be the matrix with entries

\[ c_{n,k}(N) = \begin{cases} \frac{1}{n+N} & 0 \leq k < n+N \\ 0 & k \geq n+N. \end{cases} \]

Then,

\[ L_{w,p} \left( (C^1_N)^t \right) = L_{w,p^*} \left( C^1_N \right) = p. \]

Moreover, for \( r \in \mathbb{N} \) and \( r > \max\{N - 2, \frac{1}{p} \} \), there exists a sequence \( \{x^m_N\}_m \) such that \( x^0_N = (0, ..., 0, x^{r-N+1}_r, ...) \geq 0 \), \( x^{r-N+1}_r \geq x^{r-N+2}_r \geq \ldots \)
..., \|x_N^m\|_{w,p} = 1, for all m, and also
\lim_{m \to \infty} \|x_N^m\|_{w,1} = 0, \lim_{m \to \infty} \left\| \left( C_N^1 \right)^t x_N^m \right\|_{w,p} = p.

Proof. Applying Proposition 2.5 of [6], it suffices to prove the case
\( L_{w,p} \left( \left( C_N^1 \right)^t \right) = p. \)
For \( x \geq 0 \), we have
\[ \left\| \left( C_N^1 \right)^t x \right\|_{w,p} = \left\| C(1)^t x' \right\|_{w,p}, \]
where, \( x' = \{ x'_k \}_{k=0}^{\infty} \) is defined by
\[ x'_k = \begin{cases} 0 & 0 \leq k < N - 1, \\ x_{k-N+1} & k \geq N - 1. \end{cases} \]

This implies that \( L_{w,p} \left( \left( C_N^1 \right)^t \right) \geq L_{w,p}(C(1)^t) = p. \)
For the rest of the proof, it suffices to prove the existence of \( \{ x_N^m \}_{m=0}^{\infty} \), for \( r \in \mathbb{N} \), with \( r > \max \{ N - 2, \frac{1}{p} \} \). Choose a sequence, say \( \{ \rho_m \}_{m=0}^{\infty} \), such that \( \rho_0 \leq r \) and \( \rho_m \downarrow \frac{1}{p} \). Define \( x_N^m = \{ x_k^m \}_{k=0}^{\infty} \) by
\[ x_k^m = \begin{cases} 0 & 0 \leq k < r - N + 1, \\ \phi(\rho_m)^{-1} \left( \frac{k+N-1-\rho_m}{k+N-1-r} \right) / \left( \frac{k+N-1}{r} \right) & k \geq r - N + 1, \end{cases} \]
where,
\[ \phi(t) = \left( \sum_{k=r-N+1}^{\infty} w_k \left\{ \frac{(k+N-1-t)}{(k+N-1-r)} / \left( \frac{k+N-1}{r} \right) \right\}^p \right)^{\frac{1}{p}}. \]

We have \( x_N^m = (0, ..., 0, x_{r-N+1}^m, ...) \geq 0 \), \( x_k^m \downarrow \), for all \( k \geq r - N + 1 \), and \( \|x_N^m\|_{w,p} = 1 \), for all \( m \). Applying ([7, Vol.I], p.77, Eq. (1.15)), we have
\[ \left( \frac{k+N-1-\rho_m}{k+N-1-r} \right) / \left( \frac{k+N-1}{r} \right) \sim \frac{\Gamma (r + 1)}{\Gamma (r - \rho_m + 1)} \left( \frac{k+N-1-r}{k+N-1-\rho_m} \right)^{-\rho_m}, \] as \( k \to \infty \).

Since \( \rho_m \downarrow \frac{1}{p} \) and \( \frac{1}{p} > 1 \), it follows from the monotone convergence theorem that \( \lim_{m \to \infty} \phi (\rho_m) = \infty. \) Moreover, there exists a constant \( C \) such that
\[ \limsup_{m \to \infty} \sum_{k=r-N+1}^{\infty} w_k \left\{ \left( \frac{k+N-1-\rho_m}{k+N-1-r} \right) / \left( \frac{k+N-1}{r} \right) \right\} \leq C \sum_{n=1}^{\infty} w_n n^{-1/p} < \infty. \]
So, \( \lim_{m \to \infty} \|x_m^N\|_{w,1} = 0 \). We know that \( C(1) \) is the same as the Hausdorff matrix \( H_\mu \) with \( d\mu(\theta) = d\theta \). By modifying the argument given in ([3], pp. 80-81), we can prove that

\[
\|(C_N^l)^tx_m^N\|_{w,p} = \|(1)^tx_m^N\|_{w,p} \to p, \quad \text{as} \quad m \to \infty,
\]

where, \( (x_m^N)^t \) is obtained from \( x_m^N \) by means of (2-1). This completes the proof of the lemma \( \square \)

In the following lemma, we extend Lemma 2.1 from matrix \( C_1^1 \) to general matrix \( C_N^l \), with \( l \in \mathbb{N} \).

**Lemma 2.2.** Suppose that \( 0 \leq p \leq 1, \frac{1}{p} + \frac{1}{p'} = 1 \) and \( l, N \in \mathbb{N} \). Let \( C_N^l = (c_{n,k}^{l})_{n,k \geq 0} \) be the matrix with

\[
c_{n,k}^{l} = \begin{cases} \frac{1}{n+N} & 0 \leq k < n + N - l + 1 \\ 0 & k \geq n + N - l + 1. \end{cases}
\]

Then,

\[
L_{w,p}\left((C_N^l)^t\right) = L_{w,p^*}\left(C_N^l\right) \leq p.
\]

Moreover, the following two assertions hold:

(i) For \( l \leq N \) and \( x \geq 0 \) with \( x \downarrow \), we have

\[
(2.2) \quad \|C_N^l)^t x\|_{w,p}^p \leq \|(1)^tx'\|_{w,p}^p \leq \|(C_N^l)^t x\|_{w,p}^p + \frac{p(l + 1)}{N^p} \|x\|_{w,p}^p,
\]

where, \( x' = \{x'_k\}_{k=0}^\infty \) is defined by (2.1).

(ii) There exists a sequence \( \{x_N\}_{N=0}^\infty \) such that \( x_N \geq 0, x_N \downarrow, \)

\[
\|x_N\|_{w,p} = 1, \quad \text{and also}
\]

\[
\lim_{N \to \infty} \|x_N\|_{w,1} = 0, \quad \lim_{N \to \infty} \left\| (C_N^l)^t x_N \right\|_{w,p} = p.
\]

**Proof.** For \( x \geq 0 \), \( \|C_N^l)^t x\|_{w,p}^p \leq \|(C_N^l)^t x\|_{w,p}^p \). Applying Lemma 2.1, we have

\[
L_{w,p}\left((C_1^1)^t\right) \leq L_{w,p^*}\left((C_N^l)^t\right) = p.
\]

The left side in (2.2) follows from the observation,

\[
\|(C_N^l)^t x\|_{w,p}^p \leq \|(C_N^l)^t x\|_{w,p}^p = \|(1)^tx'\|_{w,p}^p \quad (x \geq 0).
\]

Hence, to prove (i) it is suffices to show the right side of (2-2). Assume that \( l \leq N, x \geq 0 \) and \( x \downarrow \). Applying definition of \( x'_k \), we get
\[ \|C(1)^t x^t\|_{w,p} = \sum_{k=0}^{N-1} w_k \left( \sum_{n=0}^{\infty} \frac{x_n}{n+1} \right)^p + \sum_{k=N}^{\infty} w_k \left( \sum_{n=k}^{\infty} \frac{x_n}{n+1} \right)^p \]

(2.3)

\[ \leq \sum_{k=0}^{N} w_k \left( \sum_{n=0}^{\infty} \frac{x_n}{n+N} \right)^p + \sum_{k=N+1}^{\infty} w_k \left( \sum_{n=k-N+1}^{\infty} \frac{x_n}{n+N} \right)^p \]

\[ = \Sigma_1 + \Sigma_2. \]

We know that \( a^p + b^p \geq (a + b)^p \), for all \( a, b \geq 0 \). Hence,

\[ \Sigma_1 \leq \sum_{k=0}^{N-l} w_k \left( \sum_{n=0}^{\infty} c_{n,k} x_n \right)^p + \sum_{k=N-l+1}^{N} w_k \left\{ \left( \sum_{n=0}^{\infty} \frac{x_n}{n+N} \right)^p \right\} \]

(2.4)

\[ \quad + \left( \sum_{n=N-l+1}^{\infty} c_{n,k} x_n \right)^p. \]

The monotonicity of \( x_n \) implies that \( \sum_{n=0}^{k-N+l-1} \frac{x_n}{n+N} \leq \left( \frac{l}{N} \right) x_0 \), for all \( N - l < k \leq N \). Inserting this into (2.4), yields:

\[ \Sigma_1 \leq \sum_{k=0}^{N-l} w_k \left( \sum_{n=0}^{\infty} c_{n,k} x_n \right)^p + \sum_{k=N-l+1}^{N} w_k \left( \sum_{n=0}^{\infty} c_{n,k} x_n \right)^p \]

(2.5)

\[ \leq \sum_{k=N-l+1}^{N} w_k \left( \sum_{n=0}^{\infty} c_{n,k} x_n \right)^p. \]

In the same way as in (2.4), one can show

\[ \Sigma_2 \leq \sum_{k=N+1}^{\infty} w_k \left\{ \left( \sum_{n=k-N+1}^{\infty} \frac{x_n}{n+N} \right)^p \right\} \]

(2.6)

\[ \leq \frac{p}{N^p} \sum_{k=N+1}^{\infty} w_k x_k^{p} \sum_{k=N+1}^{\infty} w_k \left( \sum_{n=0}^{\infty} c_{n,k} x_n \right)^p. \]

Putting (2.3), (2.5) and (2.6) together, yields:

\[ \|C(1)^t x^t\|_{w,p} \leq \left\| \left( C_N^t \right)^t x^t \right\|_{w,p}^p + \frac{p^p (l + 1)}{N^p} \|x\|_{w,p}^p. \]

This completes the proof of (i).

(ii). Let \( x_0 = x_1 = \ldots = x_0 \), where, \( e_0 = (1, 0, 0, \ldots) \). For each \( N > \frac{1}{p} + 1 \), it follows from the case \( r = N - 1 \) of Lemma 2.1
that there exist $x_N$ with the properties: $x_N \geq 0$, $x_N \downarrow$, $\|x_N\|_{w,p} = 1$, $\|x_N\|_{w,1} \leq \frac{1}{N}$ and

$$p - \frac{1}{N} \leq \left\| \left(C_N^t \right)^t x_N \right\|_{w,p} \leq p + \frac{1}{N}.$$  

Obviously,

$$\lim_{N \to \infty} \|x_N\|_{w,1} = 0, \quad \lim_{N \to \infty} \left\| \left(C_N^t \right)^t x_N \right\|_{w,p} = p.$$  

Applying (2.2), we get

$$\left\| \left(C_N^t \right)^t x_N \right\|_{w,p}^p \leq \left\| C(1)^t x_N \right\|_{w,p}^p = \left\| (C_N^t)^t x_N \right\|_{w,p}^p \leq \left\| (C_N^t)^t x_N \right\|_{w,p}^p.$$  

Making $N \to \infty$, it follows that

$$\lim_{N \to \infty} \left\| (C_N^t)^t x_N \right\|_{w,p} = \lim_{N \to \infty} \left\| (C_N^t)^t x_N \right\|_{w,p}^p = p.$$  

This completes the proof. \hfill \Box

Note that, in general, $L_{w,p}((C_N^t)^t) \neq p$. In fact, we have $L_{w,p}((C_N^t)^t) \leq \frac{1}{N} < p$, if $N > \frac{1}{p}$. One can see this by considering the definition of $C_N^N$.

**Theorem 2.3.** Let $0 < p < 1$, $\frac{1}{p} + \frac{1}{p} = 1$ and $A = (a_{n,k})_{n,k \geq 0}$ be a lower triangular matrix with $A \geq 0$. Then,

$$(2.7) \quad pM^{p-1} \left( \inf_{n \geq 0} \sum_{k=0}^{n} a_{n,k} \right) \leq L_{w,p}(A^t).$$

Also, the same inequality holds, if $L_{w,p}(A^t)$ is replaced by $L_{w,p^*}(A)$. Here, $M$ is defined by (1.1).

**Proof.** Applying Proposition 4.3.6 of [3], we have $L_{w,p}(A^t) = L_{w,p^*}(A)$, and so it suffices to prove (2.7). Let $x \geq 0$ with $\|x\|_{w,p} = 1$. Since
$p - 1 < 0$, from Lemma 2.7.18 of [3] with (1.1) and Fubini’s theorem, it follows that:

$$\|A^t x\|_{w,p}^p = \sum_{k=0}^{\infty} w_k \left( \sum_{n=k}^{\infty} a_{n,k} x_n \right)^p$$

$$\geq p \left\{ \sum_{k=0}^{\infty} w_k \sum_{j=k}^{\infty} a_{j,k} x_j \left( \sum_{n=j}^{\infty} a_{n,j} x_n \right)^{p-1} \right\}$$

$$\geq p M^{p-1} \sum_{j=0}^{\infty} w_j x_j \left( \sum_{n=j}^{\infty} a_{n,j} x_n \right)^{p-1} a_{j,k}$$

$$\geq p M^{p-1} \left( \inf_{j \geq 0} \sum_{k=0}^{j} a_{j,k} \right) \left\{ \sum_{j=0}^{\infty} w_j x_j \left( \sum_{n=j}^{\infty} a_{n,j} x_n \right)^{p-1} \right\}.$$  

(2.8)

Applying Hölder’s inequality, we deduce that

$$\sum_{j=0}^{\infty} w_j x_j \left( \sum_{n=j}^{\infty} a_{n,j} x_n \right)^{p-1} = \sum_{j=0}^{\infty} w_j x_j \left( w_j^{\frac{p-1}{p}} \sum_{n=j}^{\infty} a_{n,j} x_n \right)^{p-1}$$

$$\geq \left( \sum_{j=0}^{\infty} w_j x_j^p \right)^{\frac{1}{p}} \left( \sum_{k=0}^{\infty} \left( w_k^{\frac{p}{p-1}} \sum_{j=0}^{k} a_{j,k} x_j \right) \right)^{\frac{p-1}{p}}$$

$$= \|x\|_{w,p} \|A^t x\|_{w,p}^{p-1}.$$  

Inserting this estimate into the corresponding term in (2.8), gives

$$\|A^t x\|_{w,p} \geq p M^{p-1} \left( \inf_{j \geq 0} \sum_{k=0}^{j} a_{j,k} \right) \|x\|_{w,p}.$$  

This leads us to the lower estimate in (2.7). \qed

Theorem 2.3 has some applications. For example, consider the weighted mean matrix, say \((A_{W_{M}}^{W_{l}})\), associated with the sequence \(W' = (w'_n)_{n=0}^{\infty}\), where, \(l = 0, 1, 2, \cdots, w'_0 = w'_1 = \cdots = w'_l = 1\) and \(w'_n = \frac{1}{2}\), for \(n > l\). Applying inequality (2.7) for \(M = 2\), we have

$$L_{w,p}(A_{W_{2}}^{W_{1}}) \geq p 2^{p-1}.$$  

Next, consider the Nörlund matrix \((A_{W}^{N_{M}})\), where, \(w' = (w'_n)_{n=0}^{\infty}\) is a non-negative sequence with \(w'_0 > 0\) and \(W'_n = \sum_{k=0}^{n} w'_k\). If \(w'_n \downarrow\), then
$M = 1$. Applying (2.7), we deduce that

$$L_{w,p}((A_W^{NW})^t) \geq p.$$ 

In general, for the summability matrix $A$ (see [1]), with increasing rows $M = 1$, we observe that (2.7) has the following form:

$$p \leq L_{w,p}(A^t) = L_{w,p}^*(A).$$

Inequality (2.9) is an analogue of ([4], Theorem 4.2), obtained by a different way.

**Theorem 2.4.** Let $0 < p < 1$, $\frac{1}{p} + \frac{1}{p'} = 1$, $w'_0 > 0$ and $w'_n \geq 0$, for all $n \geq 1$ and also $\lim_{n \to \infty} W'_n = \infty$. Then, the following assertions are true:

(i) $L_{w,p}((A_W^{NM})^t) = L_{w,p}^*(A_W^{NM}) \leq p \left( \lim_{l \to \infty} K(l) \right)$, where, $K(l) := \sup_{n \geq 0, N \geq l, l \leq k \leq n+N} \frac{(n+N+1)w'_k}{W'_{n+N}}$.

(ii) $L_{w,p}((A_W^{WM})^t) = L_{w,p}^*(A_W^{WM}) \leq p \left( \lim_{l \to \infty} k(l) \right)$, where, $k(l) := \sup_{n \geq 0, l \leq k \leq n} \frac{(n+1)w'_k}{W'_n}$.

Obviously, $k(l) \leq K(l)$, for all $l \geq 1$. Since $k(l) \downarrow$ and $K(l) \downarrow$, then the limits in (i) and (ii) can be replaced by $\inf_{l \in \mathbb{N}}$. We have

$$K(l) \leq \left( \sup_{n \geq 0} w'_n \right) / \left( \inf_{n \geq l} \frac{W'_{n+1}}{n+1} \right).$$

**Proof.** Let $x_N$ and $x'_N$ be defined as in Lemma 2.2. Since $a^p + b^p \geq (a + b)^p$, for all $a, b \geq 0$, we deduce that

$$\left\| (A_W^{NM})^t x'_N \right\|_{w,p}^p \leq \left\| (A_1^t)^t x'_N \right\|_{w,p}^p + \left\| (A_2^t)^t x'_N \right\|_{w,p}^p \quad (N \geq 0),$$

where, $A_1^t = A_W^{NM} - A_1^t$ and $A_1^t = (a_{n,k})_{n,k \geq 0}$ is the matrix obtained from $A_W^{NW}$ by replacing the $(n,k)$th entry of $A_W^{NW}$ with 0, for all $n, k$, with $n - l < k \leq n$. Consider $N \geq l + 1$. Obviously, $a_{n+N-1,k} \leq K(l)/n + N$, for $0 \leq k < n + N - l$, and $a_{n+N-1,k} = 0$, for $k \geq n + N - l$. This implies
that
\[(2.11) \quad \left\| \left( A_1^l \right)^t x_N \right\|_{w,p}^p \leq K(l)^p \left\| C_N^l \right\|^t x_N \right\|_{w,p}^p.\]

On the other hand, it follows from the definition of \(A_1^2\) that
\[(2.12) \quad \left\| \left( A_2^l \right)^t x_N \right\|_{w,p}^p \leq l \left( \frac{\max \{w'_0, w'_1, \ldots, w'_{l-1}\}}{W_{N-1}} \right)^p \|x_N\|_{w,p}^p.\]

Putting (2.10) and (2.11) together with (2.12), yields:
\[\left\| \left( A_{NM}^l W \right)^t x_N \right\|_{w,p}^p \leq K(l)^p \left\| C_N^l \right\|^t x_N \right\|_{w,p}^p + l \left( \frac{\max \{w'_0, w'_1, \ldots, w'_{l-1}\}}{W_{N-1}} \right)^p \|x_N\|_{w,p}^p.\]

We have \(\|x_N\|_{w,p} = 1\) and \(W_N \rightarrow \infty\), as \(N \rightarrow \infty\), and applying Lemma 2.2(iii), we get
\[L_{w,p}(A_{NM}^l) \leq p(\inf_{l \in \mathbb{N}} K(l)) = p \lim_{l \rightarrow \infty} K(l).\]

This proves (i).

Now, consider (ii). Let \(\{x_N^m\}_{m=0}^\infty\) be the corresponding sequence given in Lemma 2.2. Similar to \(A_{NM}^l W\), write \(A_{W}^l = A_1^l + A_2^l\), where, \(A_1^l\) is the matrix obtained from \(A_{W}^l\) by replacing the \((n,k)\)th entry of \(A_{W}^l\) with 0, for all \(n \geq 0\) and \(0 \leq k < l\). As seen above, one can easily derive:
\[\left\| \left( A_{W}^l \right)^t (x_N^m)'\right\|_{w,p}^p \leq \left\| \left( A_1^l \right)^t (x_N^m)'\right\|_{w,p}^p + \left\| \left( A_2^l \right)^t (x_N^m)'\right\|_{w,p}^p \leq (k(l))^p \left\| (C_N^l) x_N^m \right\|_{w,p}^p + l \left( \frac{\max \{w'_0, w'_1, \ldots, w'_{l-1}\}}{W_{N-1}} \right)^p \|x_N\|_{w,p}^p,
\]

which gives \(L_{w,p}(A_{W}^l) \leq pk(l),\) for all \(l \in \mathbb{N}\). Therefore,
\[L_{w,p}(A_{W}^l) \leq p(\inf_{l \in \mathbb{N}} k(l)) = p \lim_{l \rightarrow \infty} k(l).
\]

This completes the proof of the (ii).

Applying (2.9) for the summability matrix \(A\), with increasing rows, we have
\[p \leq L_{w,p}(A^t) = L_{w,p}(A)^{\ast}.
\]
Also, applying Theorem 2.4(i), we deduce the following corollaries.

**Corollary 2.5.** Let $0 < p < 1$, $\frac{1}{p} + \frac{1}{p^*} = 1$, $w_n' \downarrow \alpha$ and $\alpha > 0$. Then,

$$L_{w,p}(A_{N,M}^{\dagger}) = L_{w,p^*}(A_{N,M}^{\dagger}) = p.$$ 

**Remark 2.6.** The case $\alpha = 0$ in Corollary 2.5 is false. In general, a counterexample is the Nörlund matrix $(A_{N,M}^{W})$, where, $w_0' = 1$, $w_n' \downarrow 0$,

$$\inf_{k \geq 0} \frac{w_0'}{w_0' + \ldots + w_k} > p.$$ 

In ([5], Theorem 4.1), the upper bound of $L_{w,p}(A^t)$ is established for those summability matrices $A$, whose rows are decreasing, where, such matrices, $L_{w,p}(A^t) \leq p$. For this of type matrix, applying (2.7), we have

$$pM^{p-1} \leq L_{w,p}(A^t) \leq p.$$ 

Also, we have the following results for particular cases of such matrices.

**Corollary 2.7.** Let $0 < p < 1$, $\frac{1}{p} + \frac{1}{p^*} = 1$, $w_n' \downarrow \alpha$ and $\alpha \geq 0$. Then,

$$p(\frac{w_0'}{\alpha})^{p-1} \leq L_{w,p}(A_{N,M}^{\dagger}) = L_{w,p^*}(A_{N,M}^{\dagger}) \leq p.$$ 

**Corollary 2.8.** Let $0 < p < 1$, $\frac{1}{p} + \frac{1}{p^*} = 1$, $w_n' \downarrow \alpha$ and $w_0' > 0$. Then,

$$p(\frac{\alpha}{w_0'})^{p-1} \leq L_{w,p}(A_{N,M}^{\dagger}) = L_{w,p^*}(A_{W}^{M}) \leq p.$$ 

**References**


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