APPROSSIMATING FIXED POINTS OF GENERALIZED NONEXPANSIVE MAPPINGS

A. RAZANI AND H. SALAHIFARD

Communicated by Fraydoun Rezakhanlou

Abstract. Let $C$ be a nonempty closed convex subset of a complete $\text{CAT}(0)$ space and $T : C \to C$ be a generalized nonexpansive mapping with $F(T) = \{x \in C : T(x) = x\} \neq \emptyset$. Suppose $\{x_n\}$ is generated iteratively by $x_1 \in C,$
$$x_{n+1} = t_nT[s_nTx_n \oplus (1 - s_n)x_n] \oplus (1 - t_n)x_n,$$
for all $n \geq 1,$ where $\{t_n\}$ and $\{s_n\}$ are real sequences in $[0, 1]$ such that one of the following two conditions is satisfied:
(i) $t_n \in [a, b]$ and $s_n \in [0, 1]$, for some $a, b$ with $0 < a \leq b < 1,$
(ii) $t_n \in [a, 1]$ and $s_n \in [a, b]$, for some $a, b$ with $0 < a \leq b < 1.$

1. Introduction

Recently, Suzuki [17] introduced condition $(C)$ as follows.
Condition $(C)$: Let $T$ be a mapping on a subset $C$ of Banach space $E$.

Keywords: $\text{CAT}(0)$ spaces, Ishikawa iteration scheme, generalized nonexpansive mapping.
Received: 3 November 2010, Accepted: 4 January 2010.
*Corresponding author
© 2011 Iranian Mathematical Society.
Then, $T$ is said to satisfy condition $(C)$ (or generalized nonexpansive mapping) if
\[
\frac{1}{2}||x - Tx|| \leq ||x - y|| \text{ implies } ||Tx - Ty|| \leq ||x - y||,
\]
for all $x, y \in C$.

**Proposition 1.1.** Every nonexpansive mapping satisfies condition $(C)$, but the inverse is not true.

**Example 1.2.** Define a mapping $T$ on $[0, 3]$ by
\[
T(x) = \begin{cases} 
0 & \text{if } x \neq 3, \\
1 & \text{if } x = 3.
\end{cases}
\]
Then, $T$ satisfies condition $(C)$, but $T$ is not nonexpansive.

The purpose of this paper is to study the iterative scheme defined as follows.
Let $C$ be a nonempty closed convex subset of a complete CAT(0) space and $T : C \to C$ be a generalized nonexpansive mapping with $F(T) \neq \emptyset$.
Suppose $\{x_n\}$ is generated iteratively by $x_1 \in C$,
\begin{equation}
(1.1) \quad x_{n+1} = t_nT[s_nTx_n \oplus (1 - s_n)x_n] \oplus (1 - t_n)x_n,
\end{equation}
for all $n \geq 1$, where, $\{t_n\}$ and $\{s_n\}$ are real sequences in $[0, 1]$ such that
one of the following two conditions is satisfied:
\begin{equation}
(1.2) \quad \begin{cases} 
(ii) & t_n \in [a, b] \text{ and } s_n \in [0, 1], \text{ for some } a, b \text{ with } 0 < a \leq b < 1, \\
(iii) & t_n \in [a, 1] \text{ and } s_n \in [a, b], \text{ for some } a, b \text{ with } 0 < a \leq b < 1.
\end{cases}
\end{equation}
We show that the sequence $\{x_n\}$, defined by (1.1), $\Delta$-converges to a fixed point of $T$.

2. CAT(0) Spaces

Let $(X, d)$ be a metric space. A geodesic path joining $x \in X$ to $y \in X$ (or more briefly, a geodesic from $x$ to $y$) is a map $c$ from a closed interval $[0, l] \subset R$ to $X$ such that $c(0) = x$, $c(l) = y$, and $d(c(t), c(\hat{t})) = |t - \hat{t}|$, for all $t, \hat{t} \in [0, l]$. In particular, $c$ is an isometry and $d(x, y) = l$. The image $\alpha$ of $c$ is called a geodesic (or metric) segment joining $x$ and $y$. When it is unique, this geodesic is denoted by $[x, y]$. The space $(X, d)$ is said to be a geodesic space if every two points of $X$ are joined by a geodesic, and $X$ is said to be uniquely geodesic if there is exactly one geodesic joining
Approximating fixed points of generalized nonexpansive mappings

[132x740]Approximating fixed points of generalized nonexpansive mappings 237

x to y, for each x, y ∈ X. A subset Y ⊆ X is said to be convex if Y includes every geodesic segment joining any two of its points. A geodesic triangle △(x1, x2, x3) in a geodesic metric space (X, d) consists of three points in X (the vertices of △) and a geodesic segment between each pair of vertices (the edges of △). A comparison triangle for geodesic triangle △(x1, x2, x3) in (X, d) is a triangle △(x̄1, x̄2, x̄3) := △(x̄1, x̄2, x̄3) in the Euclidean plane E2 such that dE2(x̄i, x̄j) = d(xi, xj), for i, j ∈ {1, 2, 3}.

A geodesic metric space is said to be a CAT(0) space [1] if all geodesic triangles of appropriate size satisfy the following comparison axiom. Let △ be a geodesic triangle in X and let △ be a comparison triangle for △. Then, △ is said to satisfy the CAT(0) inequality if for all x, y ∈ △ and all comparison points x̄, ȳ ∈ △, d(x, y) ≤ dE2(x̄, ȳ). It is known that in a CAT(0) space, the distance function is convex [1]. Complete CAT(0) spaces are often called Hadamard spaces. Finally, we observe that if x, y1, y2 are points of a CAT(0) space and if y0 is the midpoint of the segment [y1, y2], which we will denote by \( \frac{y_1 + y_2}{2} \), then the CAT(0) inequality implies

\[
\begin{align*}
d(x, \frac{y_1 + y_2}{2})^2 & \leq \frac{1}{2}d(x, y_1)^2 + \frac{1}{2}d(x, y_2)^2 - \frac{1}{4}d(y_1, y_2)^2, \\
\end{align*}
\]

because equality holds in the Euclidean metric. In fact (see [1, page 163]), a geodesic metric space is a CAT(0) space if and only if it satisfies inequality (2.1) (which is known as the CN inequality of Bruhat and Tits [2]).

The following lemmas can be found in [4].

**Lemma 2.1.** Let (X, d) be a CAT(0) space. For x, y ∈ X and t ∈ [0, 1], there exists a unique point z ∈ [x, y] such that

\[ d(x, z) = td(x, y) \quad \text{and} \quad d(y, z) = (1 - t)d(x, y). \]

We use the notation \((1 - t)x \oplus ty\) for this unique z.

**Lemma 2.2.** Let (X, d) be a CAT(0) space. Then,

\[ d((1 - t)x \oplus ty, z)^2 \leq (1 - t)d(x, z)^2 + td(y, z)^2 - t(1 - t)d(x, y)^2, \]

for all t ∈ [0, 1] and x, y, z ∈ X.

The following result is of Xu [18].

**Lemma 2.3.** Let R > 1 be a fixed number and X be a Banach space. Then, X is uniformly convex if and only if there exists a continuous,
strictly increasing, and convex function $g : [0, \infty) \to [0, \infty)$ with $g(0) = 0$ such that

$$||Ax + (1 - \lambda)y||^2 \leq \lambda||x||^2 + (1 - \lambda)||y||^2 - \lambda(1 - \lambda)g(||x - y||),$$

for all $x, y \in B_{R}(0) = \{x \in X : ||x|| \leq R\}$ and $\lambda \in [0, 1]$.

Therefore, by Lemma 2.2, it turns out that $CAT(0)$ spaces offer nice examples of uniformly convex metric spaces. It is worth mentioning that the results in $CAT(0)$ spaces can be applied to any $CAT(\kappa)$ space with $\kappa \leq 0$, since any $CAT(\kappa)$ space is a $CAT(\hat{\kappa})$ space, for every $\hat{\kappa} \geq \kappa$(see [1, page 165]).

Now, we recall some definitions from [15].

Let $X$ be a complete $CAT(0)$ space and $(x_n)$ be a bounded sequence in $X$. For $x \in X$, set

$$r(x, (x_n)) = \limsup_{n \to \infty} d(x, x_n).$$

The asymptotic radius $r((x_n))$ of $(x_n)$ is given by

$$r((x_n)) = \inf \{r(x, (x_n)) : x \in X\},$$

and the asymptotic center $A((x_n))$ of $(x_n)$ is the set

$$A((x_n)) = \{x \in X : r(x, (x_n)) = r((x_n))\}.$$

**Definition 2.4.** (see [9, Definition 3.1]) A sequence $(x_n)$ in a $CAT(0)$ space $X$ is said to $\Delta$-converge to $x \in X$ if $x$ is the unique asymptotic center of $(u_n)$, for every sequence $(u_n)$ of $(x_n)$. In this case, we write $\Delta \lim_n x_n = x$ and call $x$ the $\Delta$-lim of $(x_n)$.

It is known that in a $CAT(0)$ space, $A((x_n))$ consists of exactly one point [6]. Also, every $CAT(0)$ space has the Opial property, i.e., if $(x_n)$ is a sequence in $K$ and $\Delta \lim_n x_n = x$, then for each $y(\neq x) \in K$,

$$\limsup_{n} d(x_n, x) < \limsup_{n} d(x_n, y).$$

**Lemma 2.5.** [9] Every bounded sequence in a complete $CAT(0)$ space always has a $\Delta$-convergent subsequence.

**Lemma 2.6.** [5] Let $C$ be a closed convex subset of a complete $CAT(0)$ space and $\{x_n\}$ be a bounded sequence in $C$. Then, the asymptotic center of $\{x_n\}$ is in $C$.
Lemma 2.7. [17] Let $C$ be a closed convex subset of a complete CAT(0) space $X$, and $T : C \to C$ be a generalized nonexpansive mapping. Then,
$$d(x, Ty) \leq 3d(x, Tx) + d(x, y),$$
for all $x, y \in C$.

The following result is a consequence of Lemma 2.9 in [10].

Lemma 2.8. Let $X$ be a complete CAT(0) space and $x \in X$. Suppose $\{t_n\}$ is a sequence in $[b, c]$, for some $b, c \in (0, 1)$, and $\{x_n\}, \{y_n\}$ are sequences in $X$ such that $\limsup_n d(x_n, x) \leq r$, $\limsup_n d(y_n, x) \leq r$, and $\lim_n d((1 - t_n)x_n \oplus t_n y_n, x) = r$, for some $r \geq 0$. Then,
$$\lim_{n \to \infty} d(x_n, y_n) = 0.$$

3. Main Result

Here, our main result is presented.

Theorem 3.1. Let $C$ be a nonempty closed convex subset of a complete CAT(0) space $X$ and $T : C \to C$ be a generalized nonexpansive mapping. Suppose $x_1 \in C$ and $\{x_n\}$ is defined by (1.1), where sequences $\{t_n\}, \{s_n\}$ are given by (1.2). Then, $\lim_{n \to \infty} d(x_n, x^*)$ exists, for all $x^* \in F(T)$.

Proof. Set $y_n = s_n Tx_n \oplus (1 - s_n)x_n$. Since $T$ is generalized nonexpansive and $x^* \in F(T)$,
$$\frac{1}{2} d(x^*, Tx^*) = 0 \leq d(x^*, y_n),$$
and
$$\frac{1}{2} d(x^*, Tx^*) = 0 \leq d(x^*, x_n),$$
for all $n \geq 1$. It implies $d(Tx_n, Ty_n) \leq d(x^*, y_n)$ and $d(Tx^*, Tx_n) \leq d(x^*, x_n)$. So,
$$d(x_{n+1}, x^*) = d(t_n T[s_n Tx_n \oplus (1 - s_n)x_n] \oplus (1 - t_n)x_n, x^*)$$
$$\leq t_n d(Ty_n, x^*) + (1 - t_n)d(x_n, x^*)$$
$$\leq t_n d(y_n, x^*) + (1 - t_n)d(x_n, x^*)$$
$$\leq t_n(s_n d(Tx_n, x^*) + (1 - s_n)d(x_n, x^*)) + (1 - t_n)d(x_n, x^*)$$
$$\leq d(x_n, x^*).$$

This implies $d(x_n, x^*)$ is decreasing and bounded below, and so $\lim_{n \to \infty} d(x_n, x^*)$ exists. \qed
Theorem 3.2. Let $C$ be a nonempty closed convex subset of a complete $\text{CAT}(0)$ space $X$ and $T : C \to C$ be a generalized nonexpansive mapping. From arbitrary $x_1 \in C$, define the sequence $\{x_n\}$ by (1.1), where sequences $\{t_n\}, \{s_n\}$ are given by (1.2). Then, $F(T)$ is nonempty if and only if $\{x_n\}$ is bounded and $\lim_n d(Tx_n, x_n) = 0$.

Proof. Suppose that $F(T)$ is nonempty and $x^* \in F(T)$. Then, by Theorem 3.1, $\lim_n d(x_n, x^*)$ exists and $\{x_n\}$ is bounded. Set
\begin{equation}
(3.1) \quad c = \lim_n d(x_n, x^*)
\end{equation}
and $y_n = s_nTx_n \oplus (1 - s_n)x_n$, for all $n \geq 1$. Since
\[
\frac{1}{2}d(x^*, Tx^*) = 0 \leq d(x^*, y_n),
\]
and
\[
\frac{1}{2}d(x^*, Tx^*) = 0 \leq d(x^*, x_n),
\]
for all $n \geq 1$, then $d(Tx^*, Ty_n) \leq d(x^*, y_n)$ and $d(Tx^*, Tx_n) \leq d(x^*, x_n)$.

Thus,
\[
d(Ty_n, x^*) \leq d(y_n, x^*)
\]
\[
= d(s_nTx_n \oplus (1 - s_n)x_n, x^*)
\]
\[
\leq s_n d(Tx_n, x^*) + (1 - s_n)d(x_n, x^*)
\]
\[
\leq s_n d(x_n, x^*) + (1 - s_n)d(x_n, x^*)
\]
\[
= d(x_n, x^*).
\]

Therefore,
\begin{equation}
(3.2) \quad \limsup_n d(Ty_n, x^*) \leq \limsup_n d(y_n, x^*) \leq c.
\end{equation}

Furthermore, we have
\begin{equation}
(3.3) \quad \lim_n d(t_nTy_n \oplus (1 - t_n)x_n, x^*) = \lim_n d(x_{n+1}, x^*) = c.
\end{equation}

Case 1: $0 < a \leq t_n \leq b < 1$ and $0 \leq s_n \leq b < 1$.

By (3.1), (3.2), (3.3) and Lemma 2.8, we have $\lim_n d(Ty_n, x_n) = 0$. Since for each $s_n \in [0, b]$,
\[
d(Tx_n, x_n) \leq d(Tx_n, y_n) + d(y_n, x_n)
\]
\[
\leq (1 - s_n)d(x_n, Tx_n) + d(y_n, x_n),
\]
then we have
\[
s_n d(x_n, Tx_n) \leq d(y_n, x_n).
\]
Since $T$ is generalized nonexpansive, by choosing $s_n = \frac{1}{2}$, we obtain $d(Tx_n, Ty_n) \leq d(x_n, y_n)$, and so it follows:

$$d(Tx_n, x_n) \leq d(Tx_n, Ty_n) + d(Ty_n, x_n) \leq d(x_n, y_n) + d(Ty_n, x_n) = d(s_nTx_n \oplus (1 - s_n)x_n, x_n) + d(Ty_n, x_n) \leq s_n d(Tx_n, x_n) + d(Ty_n, x_n).$$

Thus, we have $(1 - b)d(Tx_n, x_n) \leq (1 - s_n)d(Tx_n, x_n) \leq d(Ty_n, x_n)$.

Therefore, $\lim_n d(Tx_n, x_n) \leq \frac{1}{(1 - b)} \lim_n d(Ty_n, x_n) = 0$.

Case 2: $0 < a \leq t_n \leq 1$ and $0 < a \leq s_n \leq b < 1$.

Since we have $d(Tx_n, x^*) \leq d(x_n, x^*)$, for all $n \geq 1$, we get

$$\lim_n d(Tx_n, x^*) \leq c. \tag{3.4}$$

Now,

$$d(x_{n+1}, x^*) \leq t_n d(Ty_n, x^*) + (1 - t_n)d(x_n, x^*) \leq t_n d(y_n, x^*) + (1 - t_n)d(x_n, x^*) = t_n d(y_n, x^*) + d(x_n, x^*) - t_n d(x_n, x^*),$$

which implies

$$\frac{d(x_{n+1}, x^*) - d(x_n, x^*)}{t_n} \leq d(y_n, x^*) - d(x_n, x^*).$$

Taking $\liminf$ from both sides of the above inequality, we have

$$\liminf \frac{d(x_{n+1}, x^*) - d(x_n, x^*)}{t_n} \leq \liminf (d(y_n, x^*) - d(x_n, x^*)).$$

Since $\lim d(x_{n+1}, x^*) = \lim d(x_n, x^*) = c$, then

$$0 \leq \liminf (d(y_n, x^*) - d(x_n, x^*)).$$

On the other hand, since $d(y_n, x^*) - d(x_n, x^*) \leq 0$, $\liminf (d(y_n, x^*) - d(x_n, x^*)) \leq 0$. Therefore, $\liminf (d(y_n, x^*) - d(x_n, x^*)) = 0$. This shows

$$0 = \liminf (d(y_n, x^*) - d(x_n, x^*)) \leq \liminf d(y_n, x^*) - \liminf d(x_n, x^*).$$

Therefore, $\liminf d(x_n, x^*) \leq \liminf d(y_n, x^*)$. This means that $c \leq \liminf_n d(y_n, x^*)$. By combining this inequality and (3.2), we have

$$c \leq \liminf_n d(y_n, x^*) \leq \limsup_n d(y_n, x^*) \leq c. \tag{5.5}$$

Therefore,

$$c = \lim_n d(y_n, x^*) = \lim d(s_n Tx_n \oplus (1 - s_n)x_n, x^*).$$
By (3.5), (3.4), (3.1) and Lemma 2.8, we have \( \lim_{n} d(Tx_n, x_n) = 0 \).
Conversely, suppose that \( \{x_n\} \) is bounded and \( \lim_{n} d(x_n, Tx_n) = 0 \). Let 
\( A((x_n)) = \{x\} \). Then, \( x \in C \), by Lemma 2.6. Since \( T \) is generalized nonexpansive, we have, by Lemma 2.7,
\[
d(x_n,Tx) \leq 3d(x_n,Tx_n) + d(x_n,x),
\]
which implies
\[
\limsup_{n} d(x_n,Tx) \leq \limsup_{n}[3d(x_n,Tx_n) + d(x_n,x)] = \limsup_{n} d(x_n,x).
\]
By the uniqueness of asymptotic centers, we get \( Tx = x \). Therefore, \( x \) is a fixed point of \( T \).

**Theorem 3.3.** Let \( C \) be a nonempty closed convex subset of a complete \( \text{CAT}(0) \) space \( X \), and \( T : C \to C \) be a generalized nonexpansive mapping with \( F(T) \neq \emptyset \). Suppose \( \{x_n\} \) is defined by (1.1), where \( \{t_n\} \) and \( \{s_n\} \) are given by (1.2). Then, \( \{x_n\} \), \( \Delta \)-converges to a fixed point of \( T \).

**Proof.** Theorem 3.2 guarantees that \( \{x_n\} \) is bounded and
\[
\lim_{n} d(x_n,Tx_n) = 0.
\]
Let \( W_w(x_n) := \bigcup A(u_n) \), where the union is taken over all subsequences \( \{u_n\} \) of \( \{x_n\} \). We claim that \( W_w(x_n) \subset F(T) \).
Let \( u \in W_w(x_n) \). Then, there exists a subsequence \( \{u_n\} \) of \( \{x_n\} \) such that \( A((u_n)) = \{u\} \). By Lemmas 2.5 and 2.6, there exists a subsequence \( v_n \) of \( u_n \) such that \( \Delta - \lim_{n} v_n = v \in C \). Since \( \lim_{n} d(v_n,Tv_n) = 0 \) and \( T \) is generalized nonexpansive, then, by Lemma 2.7,
\[
d(v_n,Tv) \leq 3d(v_n,Tv_n) + d(v_n,v).
\]
By taking \( \lim \) and \( \text{Opial} \) property, we obtain \( v \in F(T) \). Now, we claim that \( u = v \). If not, by Theorem 3.1, \( \lim_{n} d(x_n,v) \) exists, and thus by the uniqueness of asymptotic centers,
\[
\limsup_{n} d(v_n,v) \leq \limsup_{n} d(v_n,u) = \limsup_{n} d(u_n,u) = \limsup_{n} d(v_n,v) = \limsup_{n} d(x_n,v) = \limsup_{n} d(v_n,v),
\]
which is a contradiction. So, \( u = v \in F(T) \). In order to show \( \{x_n\} \), \( \Delta \)-converges to a fixed point of \( T \), it suffices to show that \( W_w(x_n) \) consists
of exactly one point. Let \( \{u_n\} \) be a subsequence of \( \{x_n\} \). By lemmas 2.5 and 2.6, there exists a subsequence \( \{v_n\} \) of \( \{u_n\} \) such that \( \Delta - \lim_n v_n = v \in C \). Let \( A((u_n)) = \{u\} \) and \( A((x_n)) = \{x\} \). We have seen that \( v = u \) and \( v \in F(T) \). Therefore, we can complete the proof by showing that \( v = x \). If not, since \( \{d(x_n, v)\} \) is convergent by the last argument, then, by the uniqueness of asymptotic centers,

\[
\limsup_n d(v_n, v) < \limsup_n d(v_n, x) \\
\leq \limsup_n d(x_n, x) \\
< \limsup_n d(x_n, v) \\
= \limsup_n d(u_n, v),
\]

which is a contradiction, and hence the conclusion follows. \( \square \)

We recall (see [16]), a mapping \( T : C \to C \) is said to satisfy condition (I), if there exists a nondecreasing function \( f : [0, \infty] \to [0, \infty) \) with \( f(0) = 0 \) and \( f(r) > 0 \), for all \( r > 0 \), such that \( d(x, Tx) \geq f(d(x, F(T))) \), for all \( x \in C \), where, \( d(x, F(T)) = \inf_{z \in F(T)} d(x, z) \).

**Theorem 3.4.** Let \( C \) be a nonempty closed convex subset of a complete \( CAT(0) \) space \( X \), and \( T : C \to C \) be a generalized nonexpansive mapping satisfying condition (I) with \( F(T) \neq \emptyset \). Suppose \( \{x_n\} \) is defined by (1.1), where \( \{t_n\} \) and \( \{s_n\} \) are given by (1.2). Then, \( \{x_n\} \) converges strongly to some fixed point of \( T \).

**Proof.** First, we show that \( F(T) \) is closed. Let \( \{x_n\} \) be a sequence in \( F(T) \) converging to some point \( z \in C \). Since

\[
\frac{1}{2} d(x_n, Tx_n) = 0 \leq d(x_n, z),
\]

we have

\[
\limsup_n d(x_n, Tz) = \limsup_n d(Tx_n, Tz) \\
\leq \limsup_n d(x_n, z) \\
= 0.
\]

That is, \( \{x_n\} \) converges to \( Tz \). This implies \( Tz = z \). Therefore, \( F(T) \) is closed. By Theorem 3.2, we have \( \lim_{n \to \infty} d(Tx_n, x_n) = 0 \). It follows from condition (I) that

\[
\lim_{n \to \infty} f(d(x_n, F(T))) \leq \lim_{n \to \infty} d(x_n,Tx_n) = 0.
\]

Then, \( \lim_{n \to \infty} f(d(x_n, F(T))) = 0 \). Since \( f : [0, \infty] \to [0, \infty) \) is a nondecreasing function satisfying \( f(0) = 0, f(r) > 0 \), for all \( r \in (0, \infty) \),
we obtain \( \lim_{n \to \infty} d(x_n, F(T)) = 0 \). Hence, we can choose a subsequence \( \{x_{n_k}\} \) of \( \{x_n\} \) such that

\[
d(x_{n_k}, p_k) \leq \frac{1}{2^k},
\]

for all integer \( k \geq 1 \) and some sequence \( \{p_k\} \) in \( F(T) \). Again, by Theorem 3.1,

\[
d(x_{n_k+1}, p_k) \leq d(x_{n_k}, p_k) \leq \frac{1}{2^k}.
\]

Hence,

\[
d(p_{k+1}, p_k) \leq d(p_{k+1}, x_{n_{k+1}}) + d(x_{n_{k+1}}, p_k)
\leq \frac{1}{2^{k+1}} + \frac{1}{2^k}
< \frac{1}{2^{k-1}},
\]

which implies \( \{p_k\} \) is a Cauchy sequence. Since \( F(T) \) is closed, then \( \{p_k\} \) converges strongly to a point \( p \) in \( F(T) \). It is readily seen that \( \{x_{n_k}\} \) converges strongly to \( p \). Since \( \lim_n d(x_n, p) \) exists, it must be the case that \( \lim_{n \to \infty} d(x_n, p) = 0 \). \( \square \)

**Remark 3.5.** Since every nonexpansive mapping is a generalized nonexpansive mapping, one can state all the above results for nonexpansive mappings and obtain the results in [10]. Also, by setting \( s_n = 0 \), one can obtain the results in [13].

**Acknowledgments**

The first author would like to thank the School of Mathematics of the Institute for Research in Fundamental Sciences (IPM), Teheran, Iran, for supporting this research (Grant No. 89470126).

**References**


Approximating fixed points of generalized nonexpansive mappings


---

**A. Razani**

Department of Mathematics, Faculty of Science, Imam Khomeini International University, Postal code: 34149-16818, Qazvin, Iran

and

School of Mathematics, Institute for Research in Fundamental Sciences, P.O. Box 19395-5746, Tehran, Iran

Email: razani@ikiu.ac.ir
H. Salahifard
Department of Mathematics, Faculty of Science, Imam Khomeini International University, Postal code: 34149-16818, Qazvin, Iran
Email: salahifard@gmail.com