MAXIMAL SUBSETS OF PAIRWISE NON-COMMUTING ELEMENTS OF SOME FINITE \( p \)-GROUPS

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Abstract. Let \( G \) be a group. A subset \( X \) of \( G \) is a set of pairwise non-commuting elements if \( xy \neq yx \) for any two distinct elements \( x \) and \( y \) in \( X \). If \( |X| \geq |Y| \) for any other set of pairwise non-commuting elements, \( Y \) in \( G \), then \( X \) is said to be a maximal subset of pairwise non-commuting elements. Here, we determine the cardinality of a maximal subset of pairwise non-commuting elements in any non-abelian \( p \)-groups with central quotient of order less than or equal to \( p^3 \) for any prime number \( p \). As an immediate consequence, we give this cardinality for any non-abelian group of order \( p^4 \).

1. Introduction

Let \( G \) be a non-abelian group and let \( X \) be a maximal subset of pairwise non-commuting elements of \( G \). The cardinality of such a subset is denoted by \( \omega(G) \). Also, \( \omega(G) \) is the maximal clique size in the non-commuting graph of a group \( G \). Let \( Z(G) \) be the center of \( G \). The non-commuting graph of a group \( G \) is a graph with \( G \setminus Z(G) \) as the vertices and join two distinct vertices \( x \) and \( y \), whenever \( xy \neq yx \). By a famous result of Neumann [7], answering a question of Erdős, the
finiteness of $\omega(G)$ in $G$ is equivalent to the finiteness of the factor group $G/Z(G)$. Pyber [8] has shown that there is a constant $c$ such that $|G : Z(G)| \leq c^{\omega(G)}$. Chin [4] obtained upper and lower bounds for $\omega(G)$ for an extra-special $p$-group $G$, where $p$ is an odd prime number. For $p = 2$, Isaacs (see [3], p. 40) showed that $\omega(G) = 2n + 1$ for any extra-special group $G$ of order $2^{2n+1}$. Also, in [1, Lemma 4.4], it was proved that $\omega(\text{GL}(2, q)) = q^2 + q + 1$. Furthermore, in [2, Theorem 1.1], it was shown that $\omega(\text{GL}(3, q)) = q^6 + q^5 + 3q^4 + 3q^3 + q^2 - q - 1$, for $q \geq 4$, $\omega(\text{GL}(3, 2)) = 56$ and $\omega(\text{GL}(3, 3)) = 1067$. Here, we show that $\omega(G) = p + 1$, for any finite $p$-group $G$ with central quotient of order $p^2$, where $p$ is a prime number (Lemma 3.1). Also, we find $\omega(G)$, for any finite $p$-group $G$ with central quotient of order $p^3$ (Theorem 3.3). As an immediate consequence, we determine $\omega(G)$ for any non-abelian group of order $p^4$.

Throughout this paper, we use the following notation: $p$ denotes a prime number, $C_G(x)$ is the centralizer of an element $x$ in a group $G$, the nilpotency class of a group $G$ is shown by $\text{cl}(G)$, and a $p$-group of maximal class is a non-abelian group $G$ of order $p^n$ with $\text{cl}(G) = n - 1$.

2. Basic results

In this section, we give some basic results needed for any main results.

**Lemma 2.1.** Let $G$ be a finite group. Then,

(i) for any subgroup $H$ of $G$, $\omega(H) \leq \omega(G)$, and

(ii) for any normal subgroup $N$ of $G$, $\omega(G/N) \leq \omega(G)$.

**Proof.** This is evident. □

A group $G$ is called an $AC$-group, if the centralizer of every non-central element of $G$ is abelian.

**Lemma 2.2.** The followings on a group $G$ are equivalent.

(i) $G$ is an $AC$-group.

(ii) If $[x, y] = 1$, then $C_G(x) = C_G(y)$, where $x, y \in G \setminus Z(G)$.

(iii) If $[x, y] = [x, z] = 1$, then $[y, z] = 1$, where $x \in G \setminus Z(G)$.

(iv) If $A$ and $B$ are subgroups of $G$ and $Z(G) < C_G(A) \leq C_G(B) < G$, then $C_G(A) = C_G(B)$.

**Proof.** This is straightforward. See also [9], Lemma 3.2. □

**Lemma 2.3.** Let $G$ be an $AC$-group.
(i) If \( a, b \in G \setminus Z(G) \) with distinct centralizers, then \( C_G(a) \cap C_G(b) = Z(G) \).

(ii) If \( G = \bigcup_{i=1}^{k} C_G(a_i) \), where \( C_G(a_i) \) and \( C_G(a_j) \) are distinct for \( 1 \leq i < j \leq k \), then \( \{a_1 \ldots a_k\} \) is a maximal set of pairwise non-commuting elements.

**Proof.**

(i) We see that \( Z(G) \leq C_G(a) \cap C_G(b) \). If \( Z(G) < C_G(a) \cap C_G(b) \), then there exists an element \( x \) in \( C_G(a) \cap C_G(b) \) such that \( x \notin Z(G) \). This means that \( C_G(a) = C_G(x) \) and \( C_G(b) = C_G(x) \), by Lemma 2.2 (ii), which is impossible.

(ii) By Lemma 2.2 (ii), \( \{a_1, a_2, \ldots a_k\} \) is a set of pairwise non-commuting elements. Suppose to the contrary that \( \{b_1, b_2, \ldots, b_t\} \) is another set of non-commuting elements of \( G \) with \( t > k \). Then, we see that there exist positive integers \( r, s \) and \( i \) with \( r \neq s, 1 \leq r, s \leq t \) and \( 1 \leq i \leq k \), such that \( b_r, b_s \in C_G(a_i) \). This yields that \( C_G(b_r) = C_G(b_s) \), by Lemma 2.2 (ii), or equivalently \( b_r b_s = b_s b_r \), which is a contradiction. \( \square \)

### 3. Main results

In this section, we determine the cardinality of a maximal subset of pairwise non-commuting elements in any \( p \)-groups with central quotient of order less than or equal to \( p^2 \). Then, we give this cardinality for any non-abelian group of order \( p^4 \).

**Lemma 3.1.** Let \( G \) be a group of order \( p^n \) with the central quotient of order \( p^2 \), where \( p \) is a prime number. Then, \( \omega(G) = p + 1 \).

**Proof.** First, we show that \( G \) is an \( AC \)-group. Suppose that \( a \) is a non-central element of \( G \). So, \( Z(G) < C_G(a) \). Therefore, \( |C_G(a)| = p^{n-1} \). Since \( C_G(a) = \langle Z(G), a \rangle \), we see that \( C_G(a) \) is abelian and so \( G \) is an \( AC \)-group. Now, since \( G \) is finite, we may write \( G = \bigcup_{i=1}^{k} C_G(a_i) \), where \( C_G(a_i) \) and \( C_G(a_j) \) are distinct for \( 1 \leq i < j \leq k \). Therefore, \( X = \{a_1, a_2, \ldots, a_k\} \) is a maximal subset of pairwise non-commuting elements of \( G \), by Lemma 2.3 (ii). Thus, by Lemma 2.3 (i),

\[
|G| = \sum_{i=1}^{k} (|C_G(a_i)| - |Z(G)|) + |Z(G)|.
\]

This yields that \( p^n = k \times (p^{n-1} - p^{n-2}) + p^{n-2} \), and so \( k = p + 1 \). \( \square \)

**Lemma 3.2.** Let \( G \) be a group of order \( p^n \) with the central quotient of order \( p^3 \), where \( p \) is a prime number.
(i) $G$ is an AC-group.
(ii) If $G$ possesses an abelian maximal subgroup, then there exists an element $x$ in $G \setminus Z(G)$ such that $C_G(x)$ is of order $p^{n-1}$ and $C_G(x)$ is uniquely determined.

Proof. (i) Let $x \in G \setminus Z(G)$. Then, $Z(G) < Z(C_G(x)) \leq C_G(x) < G$. This yields that $|C_G(x) : Z(C_G(x))|$ divides $p$, and so $C_G(x)$ is abelian.

(ii) Let $M$ be an abelian maximal subgroup of $G$ and $x \in M \setminus Z(G)$. We see that $C_G(x) = M$, since $M \leq C_G(x) < G$. Now, if $C_G(y)$ is of order $p^{n-1}$ with $C_G(x) \neq C_G(y)$, then $C_G(x) \cap C_G(y) = Z(G)$, by Lemma 2.3 (i). Moreover, $|G : C_G(x) \cap C_G(y)| \leq |G : C_G(x)||G : C_G(y)| = p^2$, which is impossible.

\[\square\]

**Theorem 3.3.** Let $G$ be a group of order $p^n$ with the central quotient of order $p^3$, where $p$ is a prime number.

(i) If $G$ possesses no abelian maximal subgroup, then $\omega(G) = p^2 + p + 1$.
(ii) If $G$ possesses an abelian maximal subgroup, then $\omega(G) = p^2 + 1$.

Proof. (i) For any non-central element $x$ in $G$, we have $Z(G) < C_G(x) < G$. Therefore, $|C_G(x)| = p^{n-2}$, since $G$ is an AC-group. Now, we may write $G = \cup_{i=1}^k C_G(a_i)$, where $C_G(a_1)$ and $C_G(a_j)$ are distinct, for $1 \leq i < j \leq k$. Therefore, $X = \{a_1, a_2, \ldots, a_k\}$ is a maximal subset of pairwise non-commuting elements of $G$, by Lemma 2.3 (ii). Thus, by Lemma 2.3(i),

$$|G| = \sum_{i=1}^k (|C_G(a_i)| - |Z(G)|) + |Z(G)|.$$ 

This yields that $p^n = k \times (p^{n-2} - p^{n-3}) + p^{n-3}$, and so $k = p^2 + p + 1$.

(ii) By Lemma 3.2 (ii), there exists $a \in G \setminus Z(G)$ such that $C_G(a)$ is of order $p^{n-1}$ and this is the only centralizer of order $p^{n-1}$. Now, we may write $G = \cup_{i=1}^k C_G(b_i)$ such that the elements of the union are distinct. Since $a \in G$, there exists $1 \leq i \leq k$ such that $a \in C_G(b_i)$, and so $ab_i = b_i a$. Therefore, $C_G(b_i) = C_G(a)$, by Lemma 2.2 (ii). This means that $C_G(a)$ is one of the elements of the union. We may assume that $C_G(a) = C_G(b_1)$. Hence, $G = C_G(a) \cup C_G(b_2) \cup \cdots \cup C_G(b_k)$, where $|C_G(b_i)| = p^{n-2}$, for $2 \leq i \leq k$. So, by using Lemma 2.3 (i), we deduce that $|G| = |C_G(a)| + \sum_{i=2}^k (|C_G(b_i)| - |Z(G)|)$, or equivalently $p^n = p^{n-1} + (k-1)(p^{n-2} - p^{n-3})$, and hence $k = p^2 + 1$.  

\[\square\]
**Corollary 3.4.** Let $G$ be a non-abelian group of order $p^4$.

(i) If $G$ is of maximal class, then $\omega(G) = 1 + p^2$.

(ii) If $G$ is of class two, then $\omega(G) = 1 + p$.

**Proof.** (i) By Lemma 3.2, we see that $G$ is an AC-group, since $|Z(G)| = p$. Now, by considering class equation, there exists $x \in G \setminus Z(G)$ such that $|C_G(x)| = p^3$. The rest follows from Theorem 3.3 (ii).

(ii) We claim that $|Z(G)| = p^2$. For otherwise, $|Z(G)| = p$, and so, by [6, Lemma 04], we have $\exp(G/Z(G)) = \exp(G') = p$. Therefore, $G$ is an extra special group, which is a contradiction, by [10, Theorem 4.18]. Now, we can complete the proof by Lemma 3.1.

**References**


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