Title:
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Author(s):
G. Pan, L. Qiao and G. Deng
A LOWER ESTIMATE OF HARMONIC FUNCTIONS

G. PAN*, L. QIAO AND G. DENG

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ABSTRACT. We shall give a lower estimate of harmonic functions of order greater than one in a half space, which generalize the result obtained by B. Ya. Levin in a half plane.

Keywords: Lower estimate, Harmonic function, Half space.


1. Introduction

Let $\mathbb{R}$ and $\mathbb{R}_+$ be the sets of all real numbers and of all positive real numbers, respectively. Let $\mathbb{R}^n$ ($n \geq 2$) denote the $n$-dimensional Euclidean space with points $x = (x', x_n)$, where $x' = (x_1, x_2, \cdots, x_{n-1}) \in \mathbb{R}^{n-1}$ and $x_n \in \mathbb{R}$. The boundary and closure of an open set $D$ of $\mathbb{R}^n$ are denoted by $\partial D$ and $\overline{D}$, respectively. The upper half space is the set $H = \{(x', x_n) \in \mathbb{R}^n : x_n > 0\}$, whose boundary is $\partial H$.

For a set $E$, $E \subset \mathbb{R}_+ \cup \{0\}$, we denote $\{x \in H : |x| \in E\}$ and $\{x \in \partial H : |x| \in E\}$ by $HE$ and $\partial HE$, respectively. We identify $\mathbb{R}^n$ with $\mathbb{R}^{n-1} \times \mathbb{R}$ and $\mathbb{R}^{n-1}$ with $\mathbb{R}^{n-1} \times \{0\}$, writing typical points $x$, $y \in \mathbb{R}^n$ as $x = (x', x_n)$, $y = (y', y_n)$, where $y' = (y_1, y_2, \cdots, y_{n-1}) \in \mathbb{R}^{n-1}$ and putting

$$x \cdot y = \sum_{j=1}^n x_j y_j = x' \cdot y' + x_n y_n, \quad |x| = \sqrt{x \cdot x}, \quad |x'| = \sqrt{x' \cdot x'},$$

$$|x'| = |x| \cos \theta \quad \text{and} \quad x_n = |x| \sin \theta \quad (0 < \theta \leq \pi/2).$$

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*Corresponding author.
Let \( B_r \) denote the open ball with center at the origin and radius \( r > 0 \) in \( \mathbb{R}^n \). We use the standard notations \( u^+ = \max\{u, 0\} \) and \( u^- = -\min\{u, 0\} \). In the sense of Lebesgue measure \( dy' = dy_1 \cdots dy_{n-1} \) and \( dy = dy_1 dy_n \). Let \( \sigma \) denote \((n-1)\)-dimensional surface area measure and \( \partial/\partial n \) denote differentiation along the inward normal into \( H \).

The estimate we deal with has a long history which can be traced back to Levin’s estimate of harmonic functions from below (see, for example, Levin [6, p. 209]).

**Theorem 1.1.** Let \( A_1 \) be a constant, \( u(z) \) harmonic in the upper half space \( \mathbb{C}_+ \) and continuous on \( \partial\mathbb{C}_+ \). Suppose that
\[
 u(z) \leq A_1 R^\rho, \quad z \in \mathbb{C}_+, \quad R = |z| > 1, \quad \rho > 1
\]
and
\[
|u(z)| \leq A_1, \quad |z| \leq 1, \quad \text{Im} z \geq 0.
\]
Then
\[
u(\text{Re}^i\varphi) \geq -A_2 A_1 (1 + R^\rho) \sin^{-1} \varphi, \quad \text{Re}^i\varphi \in \mathbb{C}_+,
\]
where \( A_2 \) is a constant independent of \( A_1, R, \varphi \) and the function \( u(z) \).

Further versions and refinements of Theorem 1.1 may be found in the monograph Nikol’skii [7, Ch. 1] and in the paper Krasichkov-Ternovskiï [3].

In this article, we will consider functions \( u(x) \) harmonic in \( H \) and continuous on \( \overline{H} \). In what follows we shall denote by \( M \) various values which does not depend on \( K, R (=|x|), \theta \) and the function \( u(x) \).

In this note we prove analogous estimates for \( u(x) \) in \( H \).

**Theorem 1.2.** Suppose that
\[
 u(x) \leq K R^{\rho(R)}, \quad x \in H, \quad R = |x| > 1, \quad \rho(R) > 1
\]
and
\[
 u(x) \geq -K, \quad |x| \leq 1, \quad x_n \geq 0.
\]
Then
\[
u(x) \geq -MK \left( 1 + (2R)^{\rho(2R)} \right) \sin^{1-n} \theta,
\]
where \( x \in H \) and \( \rho(R) \) is nondecreasing on \([1, +\infty)\).

**Remark 1.3.** If \( n = 2 \) and \( \rho(R) \equiv \rho \), Theorem 1.2 is just a consequence of Theorem 1.1.
Theorem 1.4. If (1.1) and (1.2) hold, then
\[ u(x) \geq -MK \left( 1 + \left( \frac{N + 1}{N}R \right)^{\rho(N+1)R} \right) \sin^{1-n} \theta, \]
where \( x \in H, N(\geq 1) \) is a sufficiently large number and \( \rho(R) \) is as defined in Theorem 1.2.

2. Lemmas

Carleman’s formula [2] connects the modulus and the zeros of a function analytic in \( \mathbb{C}_+ \) (see, for example, [5, p. 224]). Nevanlinna’s formula (see [6, p. 193]) refers to a harmonic function in a half disk. Armitage and Kuran obtained a generalized Nevanlinna-type formula in a half space and Poisson integral formula for half balls respectively, which play important roles in our discussions.

Lemma 2.1. ([1]). If \( R > 1 \), then we have
\[ \int_{\{x \in H: |x| = R\}} u(x) \frac{nx_n}{R^{n+1}} d\sigma(x) + \int_{\partial H(1,R)} u(x')(\frac{1}{|x'|^n} - \frac{1}{R^n}) dx' = c_1 + \frac{c_2}{R^n}, \]
where
\[ c_1 = \int_{\{x \in H: |x| = 1\}} \left( (n-1)x_n u(x) + x_n \frac{\partial u(x)}{\partial n} \right) d\sigma(x) \]
and
\[ c_2 = \int_{\{x \in H: |x| = 1\}} \left( x_n u(x) - x_n \frac{\partial u(x)}{\partial n} \right) d\sigma(x). \]

Lemma 2.2. ([4]). Let \( R > 1 \), \( u(x) \) be a function in \( B^+_R = B_R \cap H \) and continuous in \( B^+_R \). Then
\[ u(x) = \int_{\{y \in H: |y| = R\}} \frac{R^2 - |x|^2}{\omega_n R} \left( \frac{1}{|y - x|^n} - \frac{1}{|y - x^*|^n} \right) u(y) d\sigma(y) \]
\[ + \frac{2x_n}{\omega_n} \int_{\partial H(0,R)} \left( \frac{1}{|y'|^n} - \frac{1}{|y' - \bar{x}|^n} \right) u(y') dy', \]
where \( x \in B^+_R, \bar{x} = R^2 x/|x|^2, x^* = (x', -x_n) \) and \( \omega_n = \pi^{\frac{n}{2}}/\Gamma(1 + \frac{n}{2}) \) is the volume of the unit \( n \)-ball in \( \mathbb{R}^n \).
3. Proof of Theorem 1

By applying Lemma 2.1 to \( u(x) \), we have

\[
(3.1) \quad \int_{\{x \in \mathbb{H} : |x| = R\}} u^+(x) \frac{nx_n}{R^{n+1}} |x|^n \sigma(x) + \int_{\partial H(1, R)} u^+(x') \left( \frac{1}{|x'|^n} - \frac{1}{R^n} \right) dx' \]

\[
= \int_{\{x \in \mathbb{H} : |x| = R\}} u^-(x) \frac{nx_n}{R^{n+1}} |x|^n \sigma(x) + \int_{\partial H(1, R)} u^-(x') \left( \frac{1}{|x'|^n} - \frac{1}{R^n} \right) dx' + c_1 + \frac{c_2}{R^n}.
\]

It immediately follows from (1.1) that

\[
(3.2) \quad \int_{\{x \in \mathbb{H} : |x| = R\}} u^+(x) \frac{nx_n}{R^{n+1}} |x|^n \sigma(x) \leq MKR^{\rho(R)-1}
\]

and

\[
(3.3) \quad \int_{\partial H(1, R)} u^+(x') \left( \frac{1}{|x'|^n} - \frac{1}{R^n} \right) dx' \leq MKR^{\rho(R)-1}.
\]

Hence from (3.1), (3.2) and (3.3) we have

\[
(3.4) \quad \int_{\{x \in \mathbb{H} : |x| = R\}} u^-(x) \frac{nx_n}{R^{n+1}} |x|^n \sigma(x) \leq MKR^{\rho(R)-1}
\]

and

\[
(3.5) \quad \int_{\partial H(1, R)} u^-(x') \left( \frac{1}{|x'|^n} - \frac{1}{R^n} \right) dx' \leq MKR^{\rho(R)-1}.
\]

And (3.5) gives

\[
\int_{\partial H(1, R)} \frac{u^-(x')}{|x'|^n} dx' \leq \frac{2n}{2n - 1} \int_{\partial H(1, R)} u^-(x') \left( \frac{1}{|x'|^n} - \frac{1}{(2R)^n} \right) dx' \leq MK(2R)^{\rho(2R)-1}.
\]

Since \(-u(x) \leq u^-(x)\), by applying Lemma 2.2 to \(-u(x)\) we have

\[
(3.7) \quad -u(x) \leq I_1(x) + I_2(x),
\]

where

\[
I_1(x) = \int_{\{y \in \mathbb{H} : |y| = R\}} \frac{R^2 - |x|^2}{\omega_n R} \left( \frac{1}{|y - x|^n} - \frac{1}{|y - x|^n} \right) u^-(y) \sigma(y)
\]

and

\[
I_2(x) = \int_{\{y \in \mathbb{H} : |y| = R\}} \frac{R^2 - |x|^2}{\omega_n R} \left( \frac{1}{|y - x|^n} - \frac{1}{|y - x|^n} \right) u^-(y) \sigma(y).
\]
and
\[ I_2(x) = \frac{2x_n}{\omega_n} \int_{\partial H(0,R)} \left( \frac{1}{|y' - x'|^n} - \frac{R^n}{|x|^n} \frac{1}{|y' - x|^n} \right) u^-(y') dy'. \]

We remark that
\[ \frac{1}{|y - x|^n} - \frac{1}{|x|^n} \leq \frac{2nx_n y_n}{|y - x|^{n+2}} \tag{3.8} \]
and
\[ |y - x|^n \geq x_n^n = |x|^n \sin^n \theta, \quad x \in H, \; y_n = 0. \tag{3.9} \]

If we put \(|x| = r > 1/2\) and \(R = 2r\) in (3.7), then we finally have from (3.4), (3.8) and (3.9)
\[ I_1(x) \leq \int_{\{y \in H : |y| = R\}} \frac{R^2 - r^2}{\omega_n R} \frac{2nx_n y_n}{|y - x|^{n+2}} u^-(y) d\sigma(y) \tag{3.10} \]
and
\[ I_2(x) \leq I_{21}(x) + I_{22}(x), \tag{3.11} \]
where
\[ I_{21}(x) = \frac{2}{\omega_n x_n^{n-1}} \int_{\partial H(1,R)} u^-(y') dy' \]
and
\[ I_{22}(x) = \frac{2}{\omega_n x_n^{n-1}} \int_{\partial H[0,1]} u^-(y') dy'. \]

We obtain that
\[ I_{21}(x) \leq \frac{2R^n}{\omega_n x_n^{n-1}} \int_{\partial H(1,R)} \frac{u^-(y')}{|y'|^n} dy' \tag{3.12} \]
\[ \leq MK (2R)^{\rho(2R)} \sin^{1-n} \theta \]
and
\[ I_{22}(x) \leq \frac{2K}{\omega_n x_n^{n-1}} \int_{\partial H[0,1]} dy' \tag{3.13} \]
\[ \leq MK \sin^{1-n} \theta \]
from (3.6) and (1.2), respectively.

From (3.7), (3.10), (3.11), (3.12) and (3.13), we have for \(|x| > 1/2\)
\[ -u(x) \leq MK \left( 1 + (2R)^{\rho(2R)} \right) \sin^{1-n} \theta. \tag{3.14} \]
For $|x| \leq 1/2$, we have from (1.2)

$$-u(x) \leq K \leq K \left(1 + (2R)^{\rho(2R)}\right) \sin^{1-n} \theta. \quad (3.15)$$

Thus the conclusion immediately follows from (3.14) and (3.15).

4. **Proof of Theorem 2**

By modifying (3.6), we have

$$\int_{\partial H(1,R)} \frac{u^-(x')}{|x'|^n} dx' \leq \frac{(N + 1)^n}{(N + 1)^n - N^n} \int_{\partial H(1,R)} u^-(x') \left(\frac{1}{|x'|^n} - \frac{1}{(\frac{N+1}{N}R)^n}\right) dx' \leq MK\left(\frac{N + 1}{N}R\right)^{\rho(\frac{N+1}{N}R)-1}.$$  

Then (3.12), (3.14) and (3.15) are replaced by the following estimates

$$I_{21}(x) \leq MK\left(\frac{N + 1}{N}R\right)^{\rho(\frac{N+1}{N}R)-1} \sin^{1-n} \theta. \quad (4.1)$$

$$-u(x) \leq MK\left(1 + \left(\frac{N + 1}{N}R\right)^{\rho(\frac{N+1}{N}R)}\right) \sin^{1-n} \theta. \quad (4.2)$$

$$-u(x) \leq K \leq MK\left(1 + \left(\frac{N + 1}{N}R\right)^{\rho(\frac{N+1}{N}R)}\right) \sin^{1-n} \theta. \quad (4.3)$$

All (3.7), (3.10), (3.11), (4.1), (3.12), (4.2) and (4.3) give

$$u(x) \geq -MK\left(1 + \left(\frac{N + 1}{N}R\right)^{\rho(\frac{N+1}{N}R)}\right) \sin^{1-n} \theta$$

from which the conclusion immediately follows.

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(Guoshuang Pan) Beijing National Day School, Beijing, People’s Republic of China
E-mail address: gsp1979@163.com

(Lei Qiao) College of Mathematics and Information Science, Henan University of Economics and Law, Zhengzhou, People’s Republic of China
E-mail address: qiaocqu@163.com

(Guantie Deng) School of Mathematical Science, Beijing Normal University, Laboratory of Mathematics and Complex Systems, MOE, Beijing, People’s Republic of China
E-mail address: denggt@bnu.edu.cn