Title:
n-cocoherent rings, n-cosemihereditary rings and n-V-rings

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\textbf{$n$-COCOHERENT RINGS, $n$-COSEMIHEREDITARY RINGS AND $n$-V-RINGS}

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Abstract. Let $R$ be a ring, and let $n, d$ be non-negative integers. A right $R$-module $M$ is called $(n, d)$-projective if $\text{Ext}_R^{d+1}(M, A) = 0$ for every $n$-copresented right $R$-module $A$. $R$ is called right $n$-cocoherent if every $n$-copresented right $R$-module is $(n+1)$-copresented, it is called a right co-$(n, d)$-ring if every right $R$-module is $(n, d)$-projective. $R$ is called right $n$-cosemihereditary if every submodule of a projective right $R$-module is $(n, 0)$-projective, it is called a right $n$-V-ring if it is a right co-$(n, 0)$-ring. Some properties of $(n, d)$-projective modules and $(n, d)$-projective dimensions of modules over $n$-cocoherent rings are studied. Certain characterizations of $n$-copresented modules, $(n, 0)$-projective modules, right $n$-cocoherent rings, right $n$-cosemihereditary rings, as well as right $n$-V-rings are given respectively.

Keywords: $(n, d)$-projective module; $n$-cocoherent ring; co-$(n, d)$-ring; $n$-cosemihereditary ring; $n$-V-ring.

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1. Introduction and preliminaries

Throughout this paper, $R$ is an associative ring with identity and all modules are unitary.

First we recall some known notions and facts needed in the sequel. Let $R$ be a ring, $n, d$ non-negative integers and $M$ a right $R$-module. Then:
(1) $M$ is called \textit{n-presented} \cite{1} if there is an exact sequence of right $R$-modules $F_n \to F_{n-1} \to \cdots \to F_1 \to F_0 \to M \to 0$ where each $F_i$ is a finitely generated free, equivalently projective, right $R$-module.

(2) $R$ is called \textit{right n-coherent} \cite{1} if every $n$-presented right $R$-module is $(n + 1)$-presented.

(3) $M$ is called $(n, d)$-injective \cite{13} if $\text{Ext}^{d+1}_R(A, M) = 0$ for every $n$-presented right $R$-module $A$.

(4) $R$ is called a \textit{right $(n, d)$-ring} \cite{13} if every $n$-presented right $R$-module has the projective dimension at most $d$, or equivalently, if every right $R$-module is $(n, d)$-injective. We note that a commutative right $(n, d)$-ring is called an $(n, d)$-ring in \cite{1}. Right $n$-coherent rings and right $(n, d)$-rings have been studied by several authors (see, for example, \cite{1, 2, 5, 6, 7, 13, 15}).

(5) $M$ is said to be \textit{cofree} \cite{3} if it is isomorphic to a direct product of the injective hulls of some simple right $R$-modules.

(6) $M$ is said to be \textit{finitely corelated} \cite{3} if there is a short exact sequence $0 \to M \to N \to A \to 0$ of right $R$-modules, where $N$ is finitely cogenerated, cofree, and $A$ is finitely cogenerated. It is easy to see that $M$ is finitely corelated if and only if there exists a short exact sequence of right $R$-modules $0 \to M \to E_0 \to E_1$, where each $E_i$ is a finitely cogenerated injective module. Finitely corelated modules are also called finitely copresented modules in some literatures such as \cite{10}.

(7) $M$ is said to be \textit{n-copresented} \cite{12} if there is an exact sequence of right $R$-modules $0 \to M \to E_0 \to E_1 \to \cdots \to E_n$, where each $E_i$ is a finitely cogenerated injective module.

(8) $R$ is called right \textit{co-semihereditary} \cite{8, 11, 16} if every finitely cogenerated factor module of a finitely cogenerated injective right $R$-module is injective.

(9) $R$ is called right \textit{co-coherent} (cocoherent) \cite{16} if every finitely cogenerated factor module of a finitely cogenerated injective right $R$-module is finitely copresented.

(10) $R$ is called right \textit{n-cocoherent} \cite{12} in case every $n$-copresented right $R$-module is $(n + 1)$-copresented. It is easy to see that $R$ is right cocoherent if and only if it is right 1-cocoherent. Recall that a ring $R$ is called right \textit{co-noethrian} \cite{3} if every factor module of a finitely cogenerated right $R$-module is finitely cogenerated. By \cite[Proposition 17]{3}, a ring $R$ is right co-noethrian if and only if it is right 0-cocoherent.
In this paper, we shall introduce the dual concepts of \((n, d)\)-injective right \(R\)-modules and right \((n, d)\)-rings, respectively. We shall call a right \(R\)-module \(M\) \((n, d)\)-projective if \(\text{Ext}^{d+1}_R(M, A) = 0\) for every \(n\)-copresented right \(R\)-module \(A\), and we shall call a ring \(R\) right \(co-(n, d)\)-ring if every right \(R\)-module is \((n, d)\)-projective. Some characterizations and properties of \((n, d)\)-projective modules will be provided and \((n, d)\)-projective dimensions of right \(R\)-modules over right \(n\)-cocoherent rings will be discussed. Moreover, the concepts of right \(n\)-cosemihereditary rings and right \(n\)-V-rings will be introduced and right \(n\)-cosemihereditary rings and right \(n\)-V-rings will be characterized by \((n, 0)\)-projective right \(R\)-modules.

**Lemma 1.1.** Let \(A, B\) be two right \(R\)-modules and let \(n\) be a non-negative integer. Then \(A \oplus B\) is \(n\)-copresented if and only if both \(A\) and \(B\) are \(n\)-copresented.

**Proof.** Assume that \(A\) and \(B\) are \(n\)-copresented. Then there exist two exact sequence of right \(R\)-modules

\[
0 \rightarrow A \xrightarrow{\alpha} E_0 \xrightarrow{f_0} E_1 \xrightarrow{f_1} \cdots \xrightarrow{f_{n-1}} E_n
\]

and

\[
0 \rightarrow B \xrightarrow{\beta} E'_0 \xrightarrow{g_0} E'_1 \xrightarrow{g_1} \cdots \xrightarrow{g_{n-1}} E'_n
\]

where \(E_i\) and \(E'_i\) are finitely cogenerated injective modules for all \(i\). Thus, we obtain an exact sequence of right \(R\)-modules

\[
0 \rightarrow A \oplus B \xrightarrow{\alpha \oplus \beta} E_0 \oplus E'_0 \xrightarrow{f_0 \oplus g_0} E_1 \oplus E'_1 \xrightarrow{f_1 \oplus g_1} \cdots
\]

\[
\rightarrow E_{n-1} \oplus E'_{n-1} \xrightarrow{f_{n-1} \oplus g_{n-1}} E_n \oplus E'_n
\]

where each \(E_i \oplus E'_i\) is a finitely cogenerated injective module. Thus \(A \oplus B\) is \(n\)-copresented.

Conversely, suppose that \(A \oplus B\) is \(n\)-copresented. Then there exists an exact sequence of right \(R\)-modules

\[
0 \rightarrow A \oplus B \xrightarrow{\xi} E_0 \xrightarrow{d_0} E_1 \xrightarrow{d_1} \cdots \xrightarrow{d_{n-1}} E_n
\]

where each \(E_i\) is a finitely cogenerated injective module. Hence we have an exact sequence of right \(R\)-modules

\[
0 \rightarrow A \xrightarrow{\xi} E(\xi(A)) \xrightarrow{d_{0i_0}} E(\text{Im}(d_{0i_0})) \xrightarrow{d_{1i_1}} E(\text{Im}(d_{1i_1})) \rightarrow \cdots
\]
\[
\rightarrow E(\text{Im}(d_{n-2}i_{n-2})) \rightarrow E(\text{Im}(d_{n-1}i_{n-1}))
\]

where \( E(\varepsilon(A)) \) is a direct summand of \( E_0 \), \( E(\text{Im}(d_{k}i_{k})) \) is a direct summand of \( E_{k+1} \), \( i_0 \) is the natural injection from \( E(\varepsilon(A)) \) to \( E_0 \) and \( i_k \) is the natural injection from \( E(\text{Im}(d_{k}i_{k})) \) to \( E_{k+1} \) for each \( k = 0, \cdots, n - 1 \).

Therefore, \( A \) is \( n \)-copresented. \( \square \)

Now, we give some characterizations of \( n \)-copresented modules.

**Proposition 1.2.** Let \( n \) be a positive integer. Then the following statements are equivalent for a right \( R \)-module \( M \):

1. \( M \) is \( n \)-copresented.
2. There exists an exact sequence of right \( R \)-modules
   \[
   0 \rightarrow M \rightarrow E_0 \rightarrow E_1 \rightarrow \cdots \rightarrow E_{n-1} \rightarrow L \rightarrow 0
   \]
   where \( E_0, \cdots, E_{n-1} \) are finitely cogenerated injective modules and \( L \) is finitely cogenerated.
3. \( M \) is \( (n-1) \)-copresented and, if the sequence of right \( R \)-modules
   \[
   0 \rightarrow M \rightarrow E_0 \rightarrow E_1 \rightarrow \cdots \rightarrow E_{n-1} \rightarrow L \rightarrow 0
   \]
   is exact, where \( E_0, \cdots, E_{n-1} \) are finitely cogenerated injective modules, then \( L \) is finitely cogenerated.
4. There exists an exact sequence of right \( R \)-modules
   \[
   0 \rightarrow M \rightarrow E \rightarrow L \rightarrow 0
   \]
   where \( E \) is finitely cogenerated injective and \( L \) is \( (n-1) \)-copresented.
5. \( M \) is finitely cogenerated and, if the sequence of right \( R \)-modules
   \[
   0 \rightarrow M \rightarrow E \rightarrow L \rightarrow 0
   \]
   is exact with \( E \) finitely cogenerated injective, then \( L \) is \( (n-1) \)-copresented.

**Proof.**

1. \( \Rightarrow \) (2). Since \( M \) is \( n \)-copresented, there exists an exact sequence of right \( R \)-modules
   \[
   0 \rightarrow M \rightarrow E_0 \rightarrow E_1 \rightarrow \cdots \rightarrow E_{n-1} \xrightarrow{f} E_n,
   \]
   where each \( E_i \) is finitely cogenerated injective. Let \( L = \text{Im}(f) \). Then \( L \) is finitely cogenerated and the sequence \( 0 \rightarrow M \rightarrow E_0 \rightarrow E_1 \rightarrow \cdots \rightarrow E_{n-1} \rightarrow L \rightarrow 0 \) is exact.

2. \( \Rightarrow \) (3). Follows by the dual theorem of the generalization of Schanuel’s Lemma [9, Exercise 3.37].

3. \( \Rightarrow \) (1). Since \( M \) is \( (n-1) \)-copresented, there exists an exact sequence of right \( R \)-modules
   \[
   0 \rightarrow M \rightarrow E_0 \rightarrow E_1 \rightarrow \cdots \xrightarrow{g} E_{n-1},
   \]
   where \( E_0, E_1, \cdots, E_{n-1} \) are finitely cogenerated injective modules. Let \( L = E_{n-1}/\text{Im}(g) \). Then by (3), \( L \) is finitely cogenerated. Let \( E_n = E(L) \).
Then we get an exact sequence of right $R$-modules $0 \to M \to E_0 \to E_1 \to \cdots \to E_n$ with each $E_i$ finitely cogenerated injective. Therefore, $M$ is $n$-copresented.

(1) $\Rightarrow$ (4). Since $M$ is $n$-copresented, there exists an exact sequence of right $R$-modules $0 \to M \to E \xrightarrow{\alpha} E_1 \to \cdots \to E_{n-1} \to E_n$, where $E, E_1, \cdots, E_{n-1}$ are finitely cogenerated injective modules. Let $L = \text{Im}(\alpha)$. Then it is easy to see that $L$ is $(n-1)$-copresented, and the sequence $0 \to M \to E \to L \to 0$ is exact.

(4) $\Rightarrow$ (5). Follows by the dual theorem of Schanuel’s Lemma and Lemma 1.1.

(5) $\Rightarrow$ (1). Since $M$ is finitely cogenerated, $E(M)$ is finitely cogenerated injective. By (5), $E(M)/M$ is $(n-1)$-copresented, and so there exists an exact sequence of right $R$-modules $0 \to E(M)/M \xrightarrow{h} E_1 \to \cdots \to E_{n-1} \to E_n$ with each $E_i$ finitely cogenerated injective. Thus we obtain an exact sequence of right $R$-modules $0 \to M \to E(M) \xrightarrow{h\pi} E_1 \to \cdots \to E_{n-1} \to E_n$, where $\pi$ is the natural epimorphism of $E(M)$ onto $E(M)/M$, and hence $M$ is $n$-copresented.

From Proposition 1.2(4), it is easy to see that right $n$-coherent ring is right $(n+1)$-coherent.

2. $n$-coherent rings and $(n,d)$-projective modules

We begin this section with some characterizations of right $n$-coherent rings.

Theorem 2.1. The following statements are equivalent for a ring $R$:

(1) $R$ is right $n$-coherent.

(2) If the sequence

\begin{equation}
0 \to M \xrightarrow{d_0} E_0 \xrightarrow{d_1} E_1 \xrightarrow{d_2} \cdots \xrightarrow{d_{n-1}} E_{n-1} \xrightarrow{d_n} E_n
\end{equation}

is exact, where each $E_i$ is a finitely cogenerated injective right $R$-module, then there exists an exact sequence of right $R$-modules

\begin{equation}
0 \to M \xrightarrow{d_0} E_0 \xrightarrow{d_1} E_1 \xrightarrow{d_2} \cdots \xrightarrow{d_{n-1}} E_{n-1} \xrightarrow{d_n} E_n \xrightarrow{d_{n+1}} E_{n+1}
\end{equation}

where each $E_i$ is finitely cogenerated injective.

(3) Every $(n-1)$-copresented factor module of a finitely cogenerated injective right $R$-module is $n$-copresented.
Proof. (1) ⇒ (2). By the exactness of (2.1), we have an exact sequence
\[ 0 \to M \xrightarrow{d_0} E_0 \xrightarrow{d_1} E_1 \xrightarrow{d_2} \cdots \xrightarrow{d_n} E_n \xrightarrow{d_{n+1}} E_{n+1} \to \frac{E_{n+1}}{Im(d_n)} \to 0. \]
Since \( R \) is right \( n \)-coherent, \( M \) is \((n + 1)\)-copresented. So by Proposition 1.2, \( E_{n+1}/\text{Im}(d_n) \) is finitely cogenerated. Let \( E_{n+1} = E(\frac{E_{n+1}}{\text{Im}(d_n)}) \). Then (2.2) is exact.

(2) ⇒ (1) is clear, and (1) ⇔ (3) follows by Proposition 1.2. □

Definition 2.2. Let \( n, d \) be non-negative integers. Then a right \( R \)-module \( M \) is called \((n, d)\)-projective if \( \text{Ext}^{d+1}_R(M, A) = 0 \) for every \( n \)-copresented right \( R \)-module \( A \).

Recall that a module \( M_R \) is called FCP-projective [16] if \( \text{Ext}^1_R(M, A) = 0 \) for every finitely copresented right \( R \)-module \( A \), and module \( M_R \) is called FCG-projective [14] if \( \text{Ext}^1_R(M, A) = 0 \) for every finitely cogenerated right \( R \)-module \( A \). It is obvious that \( M \) is \((0, 0)\)-projective (respectively, \((1, 0)\)-projective) if and only if \( M \) is FCG-projective (respectively, FCP-projective). For a given \( d \), every \((m, d)\)-projective module is \((n, d)\)-projective for every \( m \leq n \).

Proposition 2.3. Let \( \{M_i\}_{i \in I} \) be a family of right \( R \)-modules. Then \( \bigoplus_{i \in I} M_i \) is \((n, d)\)-projective if and only if each \( M_i \) is \((n, d)\)-projective.

Proof. Follows by the isomorphism \( \text{Ext}^{d+1}_R(\bigoplus_{i \in I} M_i, A) \cong \prod_{i \in I} \text{Ext}^{d+1}_R(M_i, A) \). □

Proposition 2.4. Let \( P \) be a projective right \( R \)-module and let \( K \) be its submodule. If \( P/K \) is \((n, d)\)-projective, then \( K \) is \((n + 1, d)\)-projective.

Proof. Let \( A \) be an \((n + 1)\)-copresented right \( R \)-module. Then there exists an exact sequence \( 0 \to A \to E \to B \to 0 \), where \( E \) is a finitely cogenerated injective module and \( B \) is \( n \)-copresented. Thus we get two exact sequences
\[
0 = \text{Ext}_R^{d+1}(P, A) \to \text{Ext}_R^{d+1}(K, A) \to \text{Ext}_R^{d+2}(P/K, A) \to \text{Ext}_R^{d+2}(P, A) = 0
\]
and
\[
0 = \text{Ext}_R^{d+1}(P/K, E) \to \text{Ext}_R^{d+1}(P/K, B) \to \text{Ext}_R^{d+2}(P/K, A) \to \text{Ext}_R^{d+2}(P/K, E) = 0.
\]
Hence \( \text{Ext}_R^{d+1}(K, A) \cong \text{Ext}_R^{d+1}(P/K, B) = 0 \), and it follows that \( K \) is \((n + 1, d)\)-projective. □
Recall that a short exact sequence of right $R$-modules $0 \to A \to B \to C \to 0$ is called copure [4] if every finitely copresented right $R$-module is injective with respect to the exact sequence, and a submodule $A$ of a right $R$-module $B$ is said to be copure in $B$ if the exact sequence $0 \to A \to B \to B/A \to 0$ is copure.

**Proposition 2.5.** Let $n \geq d+1$. Then every copure factor module of an $(n,d)$-projective module is $(n,d)$-projective. In particular, every copure factor module of an FCP-projective module is FCP-projective.

**Proof.** Let $N$ be a copure factor module of an $(n,d)$-projective module $M$. Then there exists a copure exact sequence of right $R$-modules $0 \to K \to M \to N \to 0$. For a given $n$-copresented module $A$ with a finite $n$-copresentation $0 \to A \to E_0 \to E_1 \to \cdots \to E_n$, let $L = \text{coker}(E_{d-2} \to E_{d-1})$. Then since $n \geq d+1$, $A$ is $(d+1)$-copresented, and so $L$ is finitely copresented. Since $\text{Ext}^1_R(M,L) \cong \text{Ext}^{d+1}_R(M,A) = 0$, we have an exact sequence $\text{Hom}(M,L) \xrightarrow{\partial} \text{Hom}(K,L) \xrightarrow{f^*} \text{Ext}^1_R(N,L) \to 0$. Noting that $f^*$ is epic because $N$ is a copure factor module of $M$, we have that $\partial = 0$, and hence $\text{Ext}^1_R(N,L) = 0$. Thus, $\text{Ext}^{d+1}_R(N,A) \cong \text{Ext}^1_R(N,L) = 0$, as required. \[ \square \]

**Definition 2.6.** A short exact sequence of right $R$-modules $0 \to A \to B \to C \to 0$ is called $n$-copure if every $n$-copresented right $R$-module is injective with respect to the exact sequence. A submodule $A$ of a right $R$-module $B$ is called $n$-copure in $B$ if the exact sequence $0 \to A \to B \to B/A \to 0$ is $n$-copure.

Next, we give some characterizations of $(n,0)$-projective modules.

**Theorem 2.7.** Let $n$ be a positive integer and let $M$ be a right $R$-module. Then the following statements are equivalent:

1. $M$ is $(n,0)$-projective.
2. $M$ is projective with respect to exact sequence $0 \to A \to B \to C \to 0$ of right $R$-modules with $A$ $n$-copresented.
3. If $N$ is an $(n-1)$-copresented factor module of a finitely cogenerated injective right $R$-module $E$, then every right $R$-homomorphism $f$ from $M$ to $N$ lifts to a homomorphism from $M$ to $E$.
4. Every exact sequence $0 \to M'' \to M' \to M \to 0$ is $n$-copure.
5. There exists an $n$-copure exact sequence $0 \to K \to P \to M \to 0$ of right $R$-modules with $P$ projective.
6. There exists an $n$-copure exact sequence $0 \to K \to P \to M \to 0$ of right $R$-modules with $P$ $(n,0)$-projective.
Proof. (1) $\Rightarrow$ (2). Follows by the exact sequence $\text{Hom}(M, B) \to \text{Hom}(M, C) \to \text{Ext}^1_R(M, A) = 0$.

(2) $\Rightarrow$ (3). Since the kernel of the natural epimorphism $E \to N$ is $n$-copresented, (3) follows immediately from (2).

(3) $\Rightarrow$ (1). For any $n$-copresented module $A$, there exists an exact sequence $0 \to A \to E \to N \to 0$, where $E$ is a finitely cogenerated injective module and $N$ is $(n - 1)$-copresented. So we get an exact sequence $\text{Hom}(M, E) \to \text{Hom}(M, N) \to \text{Ext}^1_R(M, A) \to \text{Ext}^1_R(M, E) = 0$, and thus $\text{Ext}^1_R(M, A) = 0$ by (3).

(1) $\Rightarrow$ (4). Assume (1). Then we have an exact sequence

$$\text{Hom}(M', A) \to \text{Hom}(M'', A) \to \text{Ext}^1_R(M, A) = 0$$

for every $n$-copresented module $A$, and so (4) follows.

(4) $\Rightarrow$ (5) $\Rightarrow$ (6) are obvious.

(6) $\Rightarrow$ (1). By (6), we have an $n$-cure exact sequence $0 \to K \xrightarrow{f} P \to M \to 0$ of right $R$-modules with $P$ $(n, 0)$-projective, and so, for each $n$-copresented module $A$, we have an exact sequence $\text{Hom}(P, A) \xrightarrow{f^*} \text{Hom}(K, A) \to \text{Ext}^1_R(M, A) \to \text{Ext}^1_R(P, A) = 0$ with $f^*$ epic. This implies that $\text{Ext}^1_R(M, A) = 0$, and (1) follows. □

**Definition 2.8.** (1). The $(n, d)$-projective dimension of a module $M_R$ is defined by

$$(n, d)-\text{pd}(M_R) = \inf\{k : \text{Ext}^{k+d+1}_R(M, A) = 0 \text{ for every } n\text{-copresented } A\}$$

(2). The right $(n, d)$-projective global dimension of a ring $R$ is defined by

$$r.(n, d)\text{-PD}(R) = \sup\{(n, d)-\text{pd}(M) : M \text{ is a right } R\text{-module}\}$$

**Lemma 2.9.** Let $R$ be a right $n$-cocoherent ring and $M$ a right $R$-module. Then the following statements are equivalent:

(1) $(n, d)-\text{pd}(M) \leq k$.

(2) $\text{Ext}^{k+d+1}_R(M, A) = 0$ for every $n$-copresented right $R$-module $A$.

Proof. (1) $\Rightarrow$ (2). Use induction on $k$. Clear if $(n, d)-\text{pd}(M) = k$. Let $(n, d)-\text{pd}(M) \leq k - 1$. Since $A$ is $n$-copresented, there exists an exact sequence $0 \to A \to E \to N \to 0$, where $E$ is a finitely cogenerated injective module and $N$ is $(n - 1)$-copresented. Since $R$ is right $n$-cocoherent, by Theorem 2.1, $N$ is $n$-copresented, and so $\text{Ext}^{k+d+1}_R(M, A) \cong \text{Ext}^{k+d}_R(M, N) = 0$ by induction hypothesis.

(2) $\Rightarrow$ (1) is clear. □
Corollary 2.10. Let $R$ be a right $n$-cocoherent ring and $M_R$ $(n, d)$-projective. Then $\text{Ext}^{d+k}_R(M, A) = 0$ for all n-copresented modules A and all positive integers $k$.

Corollary 2.11. Let $R$ be a right $n$-cocoherent ring and $M$ a right $R$-module. If the sequence $0 \to P_k \xrightarrow{f_k} P_{k-1} \xrightarrow{f_{k-1}} \cdots \to P_0 \xrightarrow{f_0} M \to 0$ is exact with $P_0, \cdots, P_{k-1}$ $(n, d)$-projective, then $\text{Ext}^{k+d+1}_R(M, A) \cong \text{Ext}_R^1(P_k, A)$ for any n-copresented right $R$-module $A$.

Proof. Since $R$ is right $n$-cocoherent and $P_0, P_1, \cdots, P_{k-1}$ are $(n, d)$-projective, by Corollary 2.10, we have

$$\text{Ext}^{k+d+1}_R(M, A) \cong \text{Ext}^{k+d}_R(\ker(f_0), A) \cong \text{Ext}^{k+d-1}_R(\ker(f_1), A) \cong \cdots \cong \text{Ext}^{d+1}_R(\ker(f_{k-1}), A) = \text{Ext}^{d+1}_R(P_k, A).$$

Theorem 2.12. Let $R$ be a right $n$-cocoherent ring, $M$ a right $R$-module and $k$ a non-negative integer. Then the following statements are equivalent:

1. $(n, d)$-pd($M_R$) $\leq k$.
2. $\text{Ext}^{k+d+l}_R(M, A) = 0$ for all n-copresented modules $A$ and all positive integers $l$.
3. $\text{Ext}^{k+d+1}_R(A, M) = 0$ for all n-copresented modules $A$.
4. If the sequence $0 \to P_k \to P_{k-1} \to \cdots \to P_0 \to M \to 0$ is exact with $P_0, \cdots, P_{k-1}$ $(n, d)$-projective, then $P_k$ is also $(n, d)$-projective.
5. There exists an exact sequence $0 \to P_k \to P_{k-1} \to \cdots \to P_0 \to M \to 0$ of right $R$-modules with $P_0, \cdots, P_{k-1}, P_k$ $(n, d)$-projective.

Proof. (1) $\Rightarrow$ (2). Assume (1), then $(n, d)$-pd($M_R$) $\leq k + l - 1$, and so (2) follows from Lemma 2.9.

(2) $\Rightarrow$ (3) and (4) $\Rightarrow$ (5) are obvious. (3) $\Rightarrow$ (4) and (5) $\Rightarrow$ (1) follow by Corollary 2.11.

3. $n$-cosemihereditary rings and $n$-$V$-rings

We set $\mu_R(M) = \sup\{n \mid M$ has a finite $n$-copresentation}, except that we set $\mu_R(M) = -1$ if $M$ is not finitely cogenerated.

Definition 3.1. Let $R$ be a ring and $n$ a non-negative integer. Then the right $n$-codimension of $R$ is defined by

$$r.n\text{-codim}(R) = \sup\{\text{id}(M_R) \mid M$ is an n-copresented right $R$-module\}
Definition 3.2. Let $R$ be a ring and $n, d$ non-negative integers. Then $R$ is said to be a right co-$(n,d)$-ring if every right $R$-module is $(n, d)$-projective.

It is easy to see that a ring $R$ is a right co-$(n,d)$-ring if and only if every $n$-copresented right $R$-module has injective dimension at most $d$ if and only if $r.n$-codim$(R) \leq d$. If $n \leq n'$ and $d \leq d'$, then every right co-$(n,d)$-ring is a right co-$(n',d')$-ring.

Lemma 3.3. Let $R$ be a ring and $M$ an $n$-copresented right $R$-module. Then $M$ is injective if and only if $\text{Ext}_1^R(A, M) = 0$ for all right $R$-modules $A$ such that $r.n$-codim$(A) \leq n - 1$.

Proof. The necessity is clear. To prove the sufficiency, let $0 \to M \to E \to N \to 0$ be exact with $E$ finitely cogenerated injective module. Then $N$ is $(n-1)$-copresented, so $\text{Ext}_1^R(N, M) = 0$ by hypothesis. It follows that $\text{Hom}_R(E, M) \to \text{Hom}_R(N, M)$ is surjective, so $M$ is isomorphic to a direct summand of $E$, and hence $M$ is injective. □

Lemma 3.4. Let $R$ be a ring and $M$ an $n$-copresented right $R$-module. Then $\text{id}(M_R) \leq d$ if and only if $\text{Ext}_1^{d+1}_R(A, M) = 0$ for all right $R$-modules $A$ such that $\mu_R(A) \geq n - (d+1)$.

Proof. The necessity is clear. The sufficiency is obvious if $d \geq n$. If $d < n$, then since $M$ is $n$-copresented, there exists an exact sequence of right $R$-modules $0 \to M \xrightarrow{\lambda} E_0 \xrightarrow{f_0} E_1 \xrightarrow{f_1} \cdots \xrightarrow{f_{d-2}} E_{d-1} \xrightarrow{f_{d-1}} \text{im}(f_{d-1}) \to 0$, where each $E_i$ is a finitely cogenerated injective module and im$(f_{d-1})$ is $(n-d)$-copresented. Thus for each right $R$-module $A$ such that $\mu_R(A) \geq n - (d+1)$, we have $\text{Ext}_1^R(A, \text{im}(f_{d-1})) \cong \text{Ext}_1^{d+1}_R(A, M) = 0$ by hypothesis. It follows that $\text{im}(f_{d-1})$ is injective. Follows by Lemma 3.3, and therefore $\text{id}(M_R) \leq d$. □

Theorem 3.5. Let $n,d \geq 1$. Then the following statements are equivalent for a ring $R$:

1. $R$ is a right co-$(n,d)$-ring.
2. $\text{Ext}_1^{d+1}_R(M, N) = 0$ for all right $R$-modules $M, N$ such that $\mu_R(N) \geq n$ and $\mu_R(M) \geq n - (d+1)$.

Proof. (1) ⇒ (2) is clear. (2) ⇒ (1) Follows by Lemma 3.4. □

Definition 3.6. A ring $R$ is called right $n$-cosemihereditary, if every submodule of a projective right $R$-module is $(n, 0)$-projective.
Clearly, a ring $R$ is right cosemihereditary if and only if it is right 1-cosemihereditary, and right $n$-cosemihereditary ring is right $(n + 1)$-cosemihe-reditary.

**Theorem 3.7.** Let $n \geq 1$. Then the following statements are equivalent for a ring $R$:

1. $R$ is a right $n$-cosemihereditary ring.
2. $R$ is a right co-$(n, 1)$-ring.
3. $\text{Ext}_R^2(M, N) = 0$ for all right $R$-modules $M, N$ such that $\mu_R(N) \geq n$ and $\mu_R(M) \geq n - 2$.
4. Every $(n-1)$-copresented factor module of a finitely cogenerated injective right $R$-module is injective.
5. $R$ is right $n$-cocoherent and $r.(n, 0)\text{-PD}(R) \leq 1$.
6. Every submodule of an $(n, 0)$-projective right $R$-module is $(n, 0)$-projective.

**Proof.** (1) $\Rightarrow$ (2). Let $A$ be any right $R$-module and $M$ any $n$-copresented right $R$-module. Then there exists an exact sequence $0 \to K \to P \to A \to 0$ with $P$ projective. By (1), $K$ is $(n, 0)$-projective, thus we have an exact sequence $0 = \text{Ext}_R^1(K, M) \to \text{Ext}_R^2(A, M) \to \text{Ext}_R^2(P, M) = 0$.

And so $\text{Ext}_R^2(A, M) = 0$, as required.

(2) $\iff$ (3). Follows by Theorem 3.5.

(2) $\Rightarrow$ (4). Let $N$ be an $(n - 1)$-copresented factor module of a finitely cogenerated injective right $R$-module $E$. Then there exists an exact sequence of right $R$-modules $0 \to K \to E \to N \to 0$. Since $K$ is $n$-copresented, by (2), $\text{Ext}_R^2(M, K) = 0$ for every right $R$-module $M$. And so $\text{Ext}_R^1(M, N) = 0$ for every right $R$-module $M$ because the sequence $0 = \text{Ext}_R^1(M, E) \to \text{Ext}_R^1(M, N) \to \text{Ext}_R^2(M, K) = 0$ is exact, as required.

(4) $\Rightarrow$ (5). By (4), every $(n - 1)$-copresented factor module of a finitely cogenerated injective right $R$-module is injective, and hence $n$-copresented, so $R$ is right $n$-cocoherent by Theorem 2.1. Now let $A$ be an $n$-copresented right $R$-module. Then we have an exact sequence $0 \to A \to E \to L \to 0$ of right $R$-modules, where $E$ is finitely cogenerated and injective, $L$ is $(n - 1)$-copresented. By hypothesis, $L$ is injective. So, for any right $R$-module $M$, the exact sequence $0 = \text{Ext}_R^1(M, L) \to \text{Ext}_R^2(M, A) \to \text{Ext}_R^2(M, E) = 0$ implies that $\text{Ext}_R^2(M, A) = 0$. It shows that $r.(n, 0)\text{-PD}(R) \leq 1$.

(5) $\Rightarrow$ (6). Let $M$ be an $(n, 0)$-projective right $R$-module and $K$ its submodule. Then for any $n$-copresented module $A$, we have an exact
sequence $0 = \text{Ext}_R^1(M, A) \to \text{Ext}_R^1(K, A) \to \text{Ext}_R^2(M/K, A) = 0$ by (5) and Lemma 2.9. It follows that $\text{Ext}_R^1(K, A) = 0$, and so $K$ is $(n, 0)$-projective.

(6) $\Rightarrow$ (1). It is obvious.

Next, we generalize the concept of right V-rings to right $n$-V-rings.

**Definition 3.8.** A ring $R$ is called right $n$-V-ring if it is a right co-$(n,0)$-ring.

Clearly, $R$ is a right V-ring if and only if it is a right 1-V-ring, and a right $n$-V-ring is a right $(n + 1)$-V-ring.

**Theorem 3.9.** The following conditions are equivalent for a ring $R$:

1. $R$ is a right $n$-V-ring.
2. Every right $R$-module is $(n, 0)$-projective.
3. Every finitely cogenerated right $R$-module is $(n, 0)$-projective.
4. $R$ is right $n$-cosemihereditary and $E(S)$ is $(n, 0)$-projective for every simple right $R$-module $S$.
5. $R$ is right $n$-cocoherent and every $n$-copresented right $R$-module is $(n, 0)$-projective.
6. Every $n$-copresented right $R$-module is injective.

**Proof.** (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are obvious.

(3) $\Rightarrow$ (4). Assume (3). Then clearly $E(S)$ is $(n, 0)$-projective. Let $E$ be a finitely cogenerated injective module and $N$ an $(n - 1)$-copresented factor module of $E$. By (3), $N$ is $(n, 0)$-projective, so by Theorem 2.7(3), $N$ is isomorphic to a direct summand of $E$ and hence $N$ is injective. Therefore, $R$ is right $n$-cosemihereditary by Theorem 3.7.

(4) $\Rightarrow$ (5). Assume (4). Since $R$ is right $n$-cosemihereditary, it is right $n$-cocoherent by Theorem 3.7. Now let $M$ be an $n$-copresented right $R$-module, then there exists an exact sequence of right $R$-modules

$$0 \to M \to E,$$

where $E$ is finitely cogenerated injective module. Since $E \cong \bigoplus_{i=1}^k E_i$ for some simple modules $E_i, i = 1, 2, \cdots, k$ and each $E_i$ is $(n, 0)$-projective, by Proposition 2.3, $E$ is $(n, 0)$-projective. Observing that $R$ is right $n$-cosemihereditary, by Theorem 3.7, $M$ is $(n, 0)$-projective.

(5) $\Rightarrow$ (6). Let $M$ be an $n$-copresented right $R$-module. Since $R$ is right $n$-cocoherent, and $M$ is $(n + 1)$-copresented, so there exists an exact sequence $0 \to M \to E \to N \to 0$ of right $R$-modules, where $E$ is finitely cogenerated injective, and $N$ is $n$-copresented. By hypothesis,
$N$ is $(n,0)$-projective, so $N$ is projective with respect to this exact sequence. This follows that $M$ is isomorphic to a direct summand of $E$, and therefore $M$ is injective.

(6) $\Rightarrow$ (1). It is clear.

\[ \square \]

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**References**


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