Title:

Stability of essential spectra of bounded linear operators

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STABILITY OF ESSENTIAL SPECTRA OF BOUNDED LINEAR OPERATORS

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Abstract. In this paper, we show the stability of Gustafson, Wei-ndmann, Kato, Wolf, Schechter and Browder essential spectrum of bounded linear operators on Banach spaces which remain invariant under additive perturbations belonging to a broad classes of operators $U$ such that $\gamma(U^m) < 1$ where $\gamma(\cdot)$ is a measure of non-compactness.

Keywords: Fredholm operators, lower (respectively, upper) semi-Fredholm operators, essential spectra, compact operators.


1. Introduction

Let $X$ and $Y$ be two Banach spaces. We denote by $\mathcal{L}(X, Y)$ the set of all bounded linear operators from $X$ into $Y$. The subspace of all compact (respectively, finite rank) operators of $\mathcal{L}(X, Y)$ is denoted by $\mathcal{K}(X, Y)$ (respectively, $\mathfrak{K}_0(X, Y)$). For $U \in \mathcal{L}(X, Y)$, we write $\mathcal{D}(U) \subset X$ for the domain, $\mathcal{N}(U) \subset X$ for the null space and $\mathcal{R}(U) \subset Y$ for the range of $U$. The nullity, $\alpha(U)$, of $U$ is defined as the dimension of $\mathcal{N}(U)$ and the deficiency, $\beta(U)$, of $U$ is defined as the codimension of $\mathcal{R}(U)$ in $Y$. The spectrum of $U$ will be denoted by $\sigma(U)$. The resolvent set $\rho(U)$ of $U$ is the complement of $\sigma(U)$ in the complex plane.

An operator $U \in \mathcal{L}(X, Y)$ is called an upper (respectively, a lower) semi-Fredholm operator, if the range $\mathcal{R}(U)$ of $U$ is closed and $\alpha(U) <$
\( \infty \) (respectively, \( \beta(U) < \infty \)). We denote by \( \Phi_+(X,Y) \) (respectively, \( \Phi_-(X,Y) \)) the set of upper (respectively, lower) semi-Fredholm operators. The set of semi-Fredholm operators is defined by \( \Phi_0(X,Y) := \Phi_+(X,Y) \cup \Phi_-(X,Y) \) and \( \Phi(X,Y) := \Phi_+(X,Y) \cap \Phi_-(X,Y) \) is called a set of Fredholm operators.

A complex number \( \lambda \) is in \( \Phi_U, \Phi_{+U}, \Phi_{-U} \) or \( \Phi_{\pm U} \), if \( \lambda - U \) is in \( \Phi(X,Y), \Phi_{+}(X,Y), \Phi_{-}(X,Y) \) or \( \Phi_{\pm}(X,Y) \), respectively. For an operator \( U \in \Phi_{\pm}(X,Y) \) we define the index of \( U \) by \( i(U) = \alpha(U) - \beta(U) \). If \( X = Y \) then \( \mathcal{L}(X,Y), \mathcal{K}(X,Y), \mathfrak{F}(X,Y), \Phi(X,Y), \Phi_{+}(X,Y), \Phi_{-}(X,Y) \) and \( \Phi_{\pm}(X,Y) \) are replaced respectively, by \( \mathcal{L}(X), \mathcal{K}(X), \mathfrak{F}(X), \Phi(X), \Phi_{+}(X), \Phi_{-}(X) \) and \( \Phi_{\pm}(X) \).

In this paper, we are concerned with the following essential spectra

\[
\sigma_{e1}(U) := \{ \lambda \in \mathbb{C} \text{ such that } \lambda - U \notin \Phi_{+}(X) \} := \mathbb{C} \setminus \Phi_{+U},
\]
\[
\sigma_{e2}(U) := \{ \lambda \in \mathbb{C} \text{ such that } \lambda - U \notin \Phi_{-}(X) \} := \mathbb{C} \setminus \Phi_{-U},
\]
\[
\sigma_{e3}(U) := \{ \lambda \in \mathbb{C} \text{ such that } \lambda - U \notin \Phi_{\pm}(X) \} := \mathbb{C} \setminus \Phi_{\pm U},
\]
\[
\sigma_{e4}(U) := \{ \lambda \in \mathbb{C} \text{ such that } \lambda - U \notin \Phi(X) \} := \mathbb{C} \setminus \Phi_U,
\]
\[
\sigma_{e5}(U) := \bigcap_{K \in \mathcal{K}(X)} \sigma(U + K),
\]
\[
\sigma_{e6}(U) := \bigcap_{U: K = KU \in \mathcal{K}(X)} \sigma(U + K).
\]

They can be ordered as

\[
\sigma_{e3}(U) := \sigma_{e1}(U) \cap \sigma_{e2}(U) \subseteq \sigma_{e4}(U) \subseteq \sigma_{e5}(U) \subseteq \sigma_{e6}(U).
\]

The subsets \( \sigma_{e1}(\cdot) \) and \( \sigma_{e2}(\cdot) \) are the Gustafson and Weidmann essential spectra \([5]\), \( \sigma_{e3}(\cdot) \) is the Kato essential spectrum \([10]\), \( \sigma_{e4}(\cdot) \) is the Wolf essential spectrum \([5, 16]\), \( \sigma_{e5}(\cdot) \) is the Schechter essential spectrum \([6, 7, 13, 15]\), and \( \sigma_{e6}(\cdot) \) denotes the Browder essential spectrum \([9, 12]\). Note that all these sets are closed and if \( X \) is a Hilbert space and \( U \) is a self-adjoint operator on \( X \), then all these sets coincide.

**Definition 1.1.** Let \( X \) and \( Y \) be two Banach spaces and let \( F \in \mathcal{L}(X,Y) \).

(i) The operator \( F \) is called a Fredholm perturbation if \( U + F \in \Phi(X,Y) \) whenever \( U \in \Phi(X,Y) \). The set of Fredholm perturbations denote by \( \mathcal{F}(X,Y) \).

(ii) The operator \( F \) is called an upper semi-Fredholm perturbation if \( U + F \in \Phi_{+}(X,Y) \) whenever \( U \in \Phi_{+}(X,Y) \). The set of upper semi-Fredholm perturbations is denoted by \( \mathcal{F}_{+}(X,Y) \).
The operator $F$ is called an lower semi-Fredholm perturbation if $U + F \in \Phi_-(X, Y)$ whenever $U \in \Phi_-(X, Y)$. The set of lower semi-Fredholm perturbations is denoted by $\mathcal{F}_-(X, Y)$.

In general we have

$$\mathcal{K}(X, Y) \subseteq \mathcal{F}_+(X, Y) \subseteq \mathcal{F}(X, Y) \quad \text{and} \quad \mathcal{K}(X, Y) \subseteq \mathcal{F}_-(X, Y) \subseteq \mathcal{F}(X, Y).$$

These classes of operators are introduced and investigated in [4]. In particular, it is shown that $\mathcal{F}(X, Y)$ is a closed subset of $\mathcal{L}(X, Y)$. If $X = Y$, we write $\mathcal{F}(X)$, $\mathcal{F}_+(X)$ and $\mathcal{F}_-(X)$ for $\mathcal{F}(X, X)$, $\mathcal{F}_+(X, X)$ and $\mathcal{F}_-(X, X)$ respectively.

We recall the following results established in [8].

**Lemma 1.2.** ([8, Lemma 2.1]) Let $U \in \mathcal{L}(X, Y)$ and $F \in \mathcal{L}(X, Y)$. Then

(i) If $U \in \Phi(X, Y)$ and $F \in \mathcal{F}(X, Y)$, then $U + F \in \Phi(X, Y)$ and $i(U + F) = i(U)$.

(ii) If $U \in \Phi_+(X, Y)$ and $F \in \mathcal{F}_+(X, Y)$, then $U + F \in \Phi_+(X, Y)$ and $i(U + F) = i(U)$.

(iii) If $U \in \Phi_-(X, Y)$ and $F \in \mathcal{F}_-(X, Y)$, then $U + F \in \Phi_-(X, Y)$ and $i(U + F) = i(U)$.

The following proposition provides a characterization of the Schechter by means of Fredholm operators.

**Proposition 1.3.** ([15, Theorem 7.27, p. 172]) Let $X$ be a Banach space and let $U \in \mathcal{L}(X)$. Then

$$\lambda \notin \sigma_{e5}(U) \text{ if and only if } \lambda \in \Phi^0_U,$$

where $\Phi^0_U = \{\lambda \in \Phi_U \text{ such that } i(\lambda - U) = 0\}$.

The notion of a measure of noncompactness is used in some problems of topology, functional analysis, and operator theory (see [3]). To recall the measure of noncompactness, we denote by $M_X$ the family of all nonempty and bounded subsets of $X$ while $N_X$ denotes its subfamily consisting of all relatively compact sets. Moreover, let conv$(A)$ denote the convex hull of a set $A \subseteq X$. In [3], a mapping $\gamma : M_X \to [0, +\infty[$ is said to be a measure of noncompactness in the space $X$, if it satisfies the following conditions:
(1) The family $\mathcal{N}(\gamma) = \{D \in M_X; \gamma(D) = 0\}$ is nonempty and $\mathcal{N}(\gamma) \subset N_X$. The family $\mathcal{N}(\gamma)$ is called the kernel of the measure of noncompactness $\gamma$.

For $S, T \in M_X$, we have the following:

(2) $\gamma(\lambda S + (1 - \lambda)T) \leq \lambda \gamma(S) + (1 - \lambda)\gamma(T)$, for all $\lambda \in [0, 1]$.

(3) If $S \subset T$ then $\gamma(S) \leq \gamma(T)$.

(4) $\gamma(S) = \gamma(S)$.

(5) $\gamma(\text{conv}(S)) = \gamma(S)$.

(6) If $(S_n)_{n \in \mathbb{N}}$ is a sequence of sets from $M_X$ such that $S_{n+1} \subset S_n$, $S_n = S_n, n \in \{1, 2, \ldots\}$ and $\lim_{n \to +\infty} \gamma(S_n) = 0$, then $S_\infty = \bigcap_{n=1}^{\infty} S_n \neq \emptyset$ and $S_\infty \in \mathcal{N}(\gamma)$.

**Definition 1.4.** (i) A measure of noncompactness $\gamma$ is said to be sublinear if for all $S, T \in M_X$, it satisfies the following two conditions:

1. $\gamma(\lambda S) = |\lambda| \gamma(S)$ for $\lambda \in \mathbb{R}$ (\gamma is said to be homogenous).
2. $\gamma(S + T) \leq \gamma(S) + \gamma(T)$ (\gamma is said to be subadditive).

(ii) A measure of noncompactness $\gamma$ is referred to as a measure with maximum property if $\max(\gamma(S), \gamma(T)) = \gamma(S \cup T)$.

(iii) A measure of noncompactness $\gamma$ is said to be regular if $\mathcal{N}(\gamma) = N_X$, it is sublinear and has maximum property.

For $S \in M_X$, the most important examples of measures of noncompactness [14] are:

- Kuratowski measure of noncompactness

$\gamma(S) = \inf\{\varepsilon > 0 : S\text{ may be covered by finitely many of sets of diameter } \leq \varepsilon\}$.

**Remark 1.5.** The Kuratowski measure of noncompactness $\gamma(\cdot)$ is regular.

Let $U \in \mathcal{L}(X)$. We say that $U$ is $k$-set-contraction if for every set $S \in M_X$, we have $\gamma(U(S)) \leq k\gamma(S)$. We define $\gamma(U)$ by

$\gamma(U) := \inf\{k : U \text{ is } k\text{-set-contraction}\}$.

We use the following proposition
Proposition 1.6. ([1, Corollary 2.3]) Let $X$ be a Banach space and $U \in \mathcal{L}(X)$. If $\gamma(U^m) < 1$ for some $m > 0$ then $I + U$ is a Fredholm operator with $i(I + U) = 0$.

We denote by $\mathcal{P}_\gamma(.)$ the set defined by

$$\mathcal{P}_\gamma(X) = \{ U \in \mathcal{L}(X) \text{ such that } \gamma(U^m) < 1, \text{ for some } m > 0 \}.$$

Definition 1.7. Let $X$ and $Y$ be two Banach spaces.

(i) An operator $U \in \mathcal{L}(X, Y)$ is said to have a left Fredholm inverse if there are maps $R_l \in \mathcal{L}(Y, X)$ and $K \in \mathcal{K}(X)$ such that $I_X + K$ extends $R_l U$. The operator $R_l$ is called a left Fredholm inverse of $U$.

(ii) An operator $U \in \mathcal{L}(X, Y)$ is said to have a right Fredholm inverse if there is a map $R_r \in \mathcal{L}(Y, X)$ such that $R_r U(\mathcal{D}(U))$ and $UR_r - I_Y \in \mathcal{K}(Y)$. The operator $R_r$ is called a right Fredholm inverse of $U$.

(iii) An operator $U \in \mathcal{L}(X,Y)$ is said to have a Fredholm inverse if we shall refer to a map which is both a left and a right Fredholm inverse of $U$.

Definition 1.8. Let $U$ and $V$ be two operators in $\mathcal{L}(X,Y)$. We denote by $F_{UV}^{\pm}(Y,X)$ the set of left or right inverses $R_{\pm}$ of $U$ satisfying $V R_+ \in \mathcal{P}_\gamma(X)$ or $R_+ V \in \mathcal{P}_\gamma(X)$ following that $U \in \Phi_{+}(X,Y)$ or $U \in \Phi_{-}(X,Y)$.

The purpose of this work is to extend the main result of Theorem 5.1 in [2] to Gustafson, Weidmann, Kato, Wolf, Schechter and Browder essential spectra of bounded linear operators on Banach spaces by means of Kuratowski measure of noncompactness where we use the set $\mathcal{P}_\gamma(.)$ as the set of perturbation operators. More precisely, let $U$ and $V$ be two operators in $\mathcal{L}(X,Y)$. If $U \in \Phi(X,Y)$ and $R$ is a Fredholm inverse of $U$ such that $VR \in \mathcal{P}_{\gamma}(X)$, then $U + V \in \Phi(X,Y)$ and $i(U + V) = i(U)$. In the same way, if $U \in \Phi_{+}(X,Y)$ (respectively, $\Phi_{-}(X,Y)$) and $R_l$ (respectively, $R_r$) is a left (respectively, right) Fredholm inverse of $U$ such that $VR_l \in \mathcal{P}_{\gamma}(Y)$ (respectively, $R_r V \in \mathcal{P}_{\gamma}(X)$), then $U + V \in \Phi_{+}(X,Y)$ (respectively, $\Phi_{-}(X,Y)$) and $i(U + V) = i(U)$. Moreover, we prove $\sigma_{ei}(U+V) \subset \sigma_{ei}(U)$, for all $i = 1, 2, 3, 4, 5, 6$ under conditions $U \lambda V$, $V U_M$ and $U_{\lambda} V \in \mathcal{P}_{\gamma}(X)$ where $U_{\lambda}$, $U_M$ and $U_{\lambda} r$ are inverse Fredholm, left Fredholm inverse and right Fredholm inverse of $\lambda - U$ respectively.

The organization of the paper is as follows: In Section 2, we present the main results.
2. Main results

**Theorem 2.1.** Let $X$ and $Y$ be two Banach spaces and let $U$ and $V$ be two operators in $\mathcal{L}(X,Y)$. Then

(i) If $U \in \Phi(X,Y)$ and $R \in \mathcal{L}(Y,X)$ is a Fredholm inverse of $U$, such that $RV \in \mathcal{P}_\gamma(X)$, then $U + V \in \Phi(X,Y)$ and $i(U + V) = i(U)$.

(ii) If $U \in \Phi_+(X,Y)$ and $R_t \in \mathcal{L}(Y,X)$ is a left Fredholm inverse of $U$, such that $VR_t \in \mathcal{P}_\gamma(X)$, then $U + V \in \Phi_+(X,Y)$ and $i(U + V) = i(U)$.

(iii) If $U \in \Phi_-(X,Y)$ and $R_r \in \mathcal{L}(Y,X)$ is a right Fredholm inverse of $U$, such that $R_rV \in \mathcal{P}_\gamma(X)$, then $U + V \in \Phi_-(X,Y)$ and $i(U + V) = i(U)$.

(iv) If $U \in \Phi_{\pm}(X,Y)$ and $F_{UV}^\pm(Y,X) \neq \emptyset$, then $U + V \in \Phi_{\pm}(X,Y)$.

**Proof.** (i) Since $R$ is a Fredholm inverse of $U$, there exists $K \in \mathcal{K}(Y)$ such that

$$UR = I - K \text{ on } Y.$$   

It follows from Eq. (2.1) that the operator $U + V$ can be written in the form

$$U + V = U + (UR + K)V = U(I_X + RV) + KV.$$  

Using the fact that $RV \in \mathcal{P}_\gamma(X)$ together with Proposition 1.6 one gets

$$I_X + RV \in \Phi(X) \text{ and } i(I_X + RV) = 0.$$  

Since $U \in \Phi(X,Y)$, applying [11, Theorem 5 (i), p. 159], we have $U(I_X + RV) \in \Phi(X,Y)$. Moreover, since $KV \in \mathcal{K}(X,Y)$, using Lemma 1.2 (i) and Eq. (2.2), we infer

$$U + V \in \Phi(X,Y) \text{ and } i(U + V) = i(U).$$  

(ii) If $R_t$ is a left Fredholm inverse of $U$, then there exists $K \in \mathcal{K}(X)$ such that

$$R_tU = I - K \text{ on } X.$$  

It follows from Eq. (2.4) that the operator $U + V$ can be written in the form

$$U + V = U + V(R_tU + K) = (VR_t + I_Y)V + VK.$$  

Using the fact that $VR_t \in \mathcal{P}_\gamma(Y)$, and applying Proposition 1.6 we have $VR_t + I_Y \in \Phi(Y)$ and $i(VR_t + I_Y) = 0$. Moreover, $U \in \Phi_+(X,Y)$, using
[11, Theorem 5 (ii), p. 156], we obtain $(VR_l + I_Y)U \in \Phi_+(X, Y)$. Since $VK \in \mathcal{K}(X, Y)$, applying Lemma 1.2 (ii) and Eq. (2.5), we get

$$U + V \in \Phi_+(X, Y) \text{ and } i(U + V) = i(U).$$

(iii) If $R_r$ is a right Fredholm inverse of $U$, then there exists $K \in \mathcal{K}(Y)$ such that

$$UR_r = I - K \text{ on } Y,$$

and consequently,

$$U + V = U + V(UR_r + K) = U(UR_r + I_X) + KV.$$

Now, arguing as in (ii) we get

$$U + V \in \Phi_-(X, Y) \text{ and } i(U + V) = i(U).$$

(iv) The statement (iv) is an immediate consequence of the items (ii) and (iii).

□

**Theorem 2.2.** Let $X$ be a Banach space and let $U$ and $V$ be two operators in $\mathcal{L}(X)$. Then the following statements hold:

(i) Assume that for each $\lambda \in \Phi_U$, there exists a Fredholm inverse $U_\lambda$ of $\lambda - U$ such that $U_\lambda V \in \mathcal{P}_\gamma(X)$, then

$$\sigma_{e4}(U + V) \subseteq \sigma_{e4}(U) \text{ and } \sigma_{e5}(U + V) \subseteq \sigma_{e5}(U).$$

(ii) If the hypotheses of (i) is satisfied and if $C\sigma_{e5}(U)$ and $C\sigma_{e5}(U+V)$ are connected, then

$$\sigma_{e6}(U + V) \subseteq \sigma_{e6}(U).$$

(iii) Assume that for each $\lambda \in \Phi_{+U}$, there exists a left Fredholm inverse $U_\lambda$ of $\lambda - U$ such that $VU_\lambda \in \mathcal{P}_\gamma(X)$, then

$$\sigma_{e1}(U + V) \subseteq \sigma_{e1}(U).$$

(iv) Assume that for each $\lambda \in \Phi_{-U}$, there exists a right Fredholm inverse $U_\lambda$ of $\lambda - U$ such that $U_\lambda V \in \mathcal{P}_\gamma(X)$, then

$$\sigma_{e2}(U + V) \subseteq \sigma_{e2}(U).$$

(v) Assume that for each $\lambda \in \Phi_{\pm U}$, the set $F^\pm_{(\lambda - U)V}(X) \neq \emptyset$, then

$$\sigma_{e3}(U + V) \subseteq \sigma_{e3}(U).$$
Proof. (i) Suppose that $\lambda \not\in \sigma_{e4}(U)$ (respectively, $\lambda \not\in \sigma_{e5}(U)$), then $\lambda \in \Phi_U$ (respectively, by Proposition 1.3, we have $\lambda \in \Phi_U$ and $i(\lambda - U) = 0$). Applying Theorem 2.1 (i) to the operators $\lambda - U$ and $-V$, we prove that $\lambda \in \Phi_{U+V}$ and $i(\lambda - U) = i(\lambda - U - V)$. Therefore $\lambda \not\in \sigma_{e4}(U + V)$ (respectively, $\lambda \not\in \sigma_{e5}(U + V)$). We obtain
\[ \sigma_{e4}(U + V) \subseteq \sigma_{e4}(U) \]
and
\[ \sigma_{e5}(U + V) \subseteq \sigma_{e5}(U). \]  

(ii) The sets $C\sigma_{e5}(U + V)$ and $C\sigma_{e5}(U)$ are connected. Since $U$ and $V$ are bounded operators, we have $\rho(U)$ and $\rho(U + V)$ are not empty sets. So, using [7, Lemma 3.1], we deduce that
\[ \sigma_{e5}(U + V) = \sigma_{e6}(U + V) \text{ and } \sigma_{e5}(U) = \sigma_{e6}(U). \]
So, Eq. (2.6) gives
\[ \sigma_{e6}(U + V) \subseteq \sigma_{e6}(U). \]

(iii) Suppose that $\lambda \notin \sigma_{e1}(U)$ then $\lambda \in \Phi_{U+V}$. Using Theorem 2.1 (ii) to the operators $\lambda - U$ and $-V$, we prove that $\lambda \in \Phi_{+/(U+V)}$. This proves that $\lambda \notin \sigma_{e1}(U + V)$. We find
\[ \sigma_{e1}(U + V) \subseteq \sigma_{e1}(U). \]

(iv) By a similar proof as in (iii), we replace $\sigma_{e1}(.)$ and $\Phi_{+}(.)$ by $\sigma_{e2}(.)$ and $\Phi_{-}(.)$ respectively, and using Theorem 2.1 (iii) we obtain
\[ \sigma_{e2}(U + V) \subseteq \sigma_{e2}(U). \]

(v) Let $\lambda \notin \sigma_{e3}(U)$ then $\lambda \in \Phi_{\pm U}$. Since $\mathcal{P}_{(\lambda - U)V}(X) \neq \emptyset$, applying Theorem 2.1 (iv) to the operators $\lambda - U$ and $-V$ we have $\lambda \in \Phi_{\pm(U+V)}$. Therefore
\[ \sigma_{e3}(U + V) \subseteq \sigma_{e3}(U). \]
\[ \square \]

Remark 2.3. (i) The results of the Theorem 2.1 remains valid if we suppose that $U \in \mathcal{C}(X)$ and $V$ is a $U$-bounded operator on $X$. Clearly, applying Theorem 2.1, we prove the statements for $\hat{U} \in \mathcal{L}(X_U, X)$ and $\hat{V} \in \mathcal{L}(X_U, X)$ and applying Eq. (2.7) we conclude the desired results.
Let $V$ be an arbitrary $U$-bounded operator, hence we can regard $U$ and $V$ as operators from $X_U$ into $X$, denoted by $\hat{U}$ and $\hat{V}$ respectively, that belong to $L(X_U, X)$. Furthermore, we have the obvious relations

\begin{equation}
\begin{aligned}
\alpha(\hat{U}) &= \alpha(U), & \beta(\hat{U}) &= \beta(U), & R(\hat{U}) &= R(U), \\
\alpha(\hat{U} + \hat{V}) &= \alpha(U + V), & \beta(\hat{U} + \hat{V}) &= \beta(U + V) \quad \text{and} \quad R(\hat{U} + \hat{V}) &= R(U + V).
\end{aligned}
\end{equation}

(ii) similarly, as in Theorem 2.2, one may show that the same results hold if we suppose that $U \in C(X)$ and assume that $V$ is an $U$-bounded operator on $X$.

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