Title:
The finite $S$–determinacy of singularities in positive characteristic, $S = R_G, R_A, K_G, K_A$

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THE FINITE $S$–DETERMINACY OF SINGULARITIES
IN POSITIVE CHARACTERISTIC, $S = \mathcal{R}_G, \mathcal{R}_A, \mathcal{K}_G, \mathcal{K}_A$

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Abstract. For singularities $f \in K[[x_1, \ldots, x_n]]$ over an algebraically closed field $K$ of arbitrary characteristic, we introduce the finite $S$–determinacy under $S$–equivalence, where $S = \mathcal{R}_G, \mathcal{R}_A, \mathcal{K}_G, \mathcal{K}_A$. It is proved that the finite $\mathcal{R}_G(\mathcal{K}_G)$–determinacy is equivalent to the finiteness of the relative $G$–Milnor ($G$–Tjurina) number and the finite $\mathcal{R}_A(\mathcal{K}_A)$–determinacy is equivalent to the finiteness of the relative $A$–Milnor ($A$–Tjurina) number. Moreover, some estimates are provided on the degree of the $S$–determinacy in positive characteristic.

Keywords: Finite $\mathcal{R}_G$ ($\mathcal{R}_A$)–determinacy, finite $\mathcal{K}_G$ ($\mathcal{K}_A$)–determinacy, the relative $G(A)$–Milnor number, relative $G(A)$–Tjurina number.


1. Introduction

In this paper, we assume that $K$ is an algebraically closed field of arbitrary characteristic unless otherwise stated explicitly. Let

$$K[[x]] = K[[x_1, \ldots, x_n]] = \left\{ \sum_{\alpha \in \mathbb{N}^n} a_{\alpha} x^{\alpha} | a_{\alpha} \in K \right\}$$

be the formal power series ring over $K$. We use the usual multi-index notation $x^{\alpha} = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ for $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$. We denote $\mathcal{M}$ =
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Let $S$ be a subgroup of $\text{Aut}(K[[x]])$. Then an equivalence relation can be introduced on $K[[x]]$ via $S$. For the given equivalence relation, a fundamental question is: when is a function $f \in K[[x]]$ equivalent to a finite number of terms of its power series. This question is concerned with the finite determinacy theory and the classification theory for map-germs.

If $K$ is the field of complex numbers and $K[[x]]$ is the ring of formal power series defined by the convergent ones, this question is well studied by John Mather and some authors (see, e.g. [1, 2, 4–6, 11–15, 17]). In the complex case, let $\mathcal{O}_{n+1,0}$ be the local ring of analytic function germs on analytic space $(\mathbb{C}^{n+1}, 0)$. Let $\{y_1, \ldots, y_{n+1}\}$ be a coordinate system in $\mathbb{C}^{n+1}$ and $\mathcal{M}$ be the maximal ideal of $\mathcal{O}_{n+1,0}$. Let $\mathcal{R}$ be the group of all the holomorphic automorphisms of the germ $(\mathbb{C}^{n+1}, 0)$. Take $L$ as the $y_1$—axis in $(\mathbb{C}^{n+1}, 0)$, then the defining ideal of $L$ is $\mathcal{G} = \langle y_2, \ldots, y_{n+1} \rangle$.

Let

$$\mathcal{R}_L = \{ \phi \in \mathcal{R} \mid \phi(L) = L \},$$

be the subgroup of the holomorphic automorphisms $\phi : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}^{n+1}, 0)$ such that $\phi(L) = L$ for all $\phi \in \mathcal{R}$. $\mathcal{R}_L$ can act on $\mathcal{M} \cdot \mathcal{G}$ from right and this defines an equivalence relation on $\mathcal{M} \cdot \mathcal{G}$. Two germs $f$, $g \in \mathcal{M} \cdot \mathcal{G}$ are called $\mathcal{R}_L$—equivalent if there exists a $\phi \in \mathcal{R}_L$ such that $f = g \circ \phi$. A germ $f \in \mathcal{M} \cdot \mathcal{G}$ is called $k$—$\mathcal{R}_L$—determined in $\mathcal{M} \cdot \mathcal{G}$ if for each $g \in \mathcal{M} \cdot \mathcal{G}$ such that $f - g \in M^{k+1} \cap \mathcal{G} = M^k \cdot \mathcal{G}$, $g$ is $\mathcal{R}_L$—equivalent to $f$.

Siersma studied the problem of finite $\mathcal{R}_L$—determinacy in [16]. He gave the list of $\mathcal{R}_L$—simple singularities and studied the Milnor fiber of a generic deformation of a certain class of such singularities.

Jiang and Siersma proved the following theorem (see Theorem 2.2. of [9]):

If $M^k \cdot \mathcal{G} \subset M \cdot \tau_\mathcal{G}(f) + M^{k+1} \cdot \mathcal{G}$, then $f$ is $k$—$\mathcal{R}_L$—determined, where

$$\tau_\mathcal{G}(f) = M \cdot \langle \frac{\partial f}{\partial y_1} \rangle + \mathcal{G} \cdot \langle \frac{\partial f}{\partial y_2}, \ldots, \frac{\partial f}{\partial y_{n+1}} \rangle$$

is the tangent space at $f$ of the $\mathcal{R}_L$—orbit $\mathcal{R}_L(f)$.

In [4], When $(X, 0)$ is the germ of an analytic subvariety of $(\mathbb{C}^n, 0)$, let $\mathcal{R}_X$ be the group of all analytic automorphisms of $(\mathbb{C}^n, 0)$ which preserve $X$. $\mathcal{R}_X$ can act on $\mathcal{O}_{n,0}$ and induce an equivalence relation. If

$$(x_1, \ldots, x_n)$$

the unique maximal ideal of $K[[x]]$, so that the set of units in $K[[x]]$ is $K[[x]]^* = K[[x]] \setminus \mathcal{M}$.
$f$ is again a function germ on $\mathbb{C}^n$ at 0. Bruce and Roberts generalized the definition of Milnor number $\mu(f)$ as follows. Let $\Theta_{X,0}$ denote the $\mathcal{O}_{n,0}$ module of germs of vector fields on $\mathbb{C}^n$ at 0 which are tangent to $X$, or equivalently, the submodule of germs of derivations of $\mathcal{O}_{n,0}$ which preserve the ideal defining $X$. For an $f \in \mathcal{O}_{n,0}$ define $j_X(f)$ the ideal in $\mathcal{O}_{n,0}$ given by the image of the homomorphism

$$\Theta_{X,0} \to \mathcal{O}_{n,0}, \delta \mapsto \delta f,$$

and define the Milnor number $\mu_X(f)$ of $f$ on $X$ to be $\dim \mathbb{C} \mathcal{O}_{n,0}/j_X(f)$. Bruce and Roberts stated Damon’s result as (see Theorem 2.2. of [4]):

A germ $f$ in $\mathcal{O}_{n,0}$ is finitely determined with respect to the $\mathcal{R}_X$ action if $\mu_X(f) < \infty$.

In [3], Yousra Boubakri, Gert-Martin Greuel, and Thomas Markwig studied the finite determinacy of singularities $f \in K[[x]]$ over an algebraically closed field $K$ of arbitrary characteristic under the equivalence relation on the power series ring $K[[x]]$ induced by the action of either $\mathcal{R} = \text{Aut}(K[[x]])$ or the semidirect product $\mathcal{K} = K[[x]]^* \rtimes \mathcal{R}$. For an $f \in K[[x]]$, they established that the finiteness of the Milnor number and the Tjurina number is equivalent to the finite $\mathcal{R}$–determinacy of $f$ and the finite $\mathcal{K}$–determinacy of $f$ respectively. The Milnor number $\mu(f)$ is defined as $\dim_K K[[x]]/j(f)$ where $j(f)$ is the Jacobian ideal of $f$, generated by the partial derivatives $f_{x_i}$ of $f$, $(i = 1, \ldots, n)$. The Tjurina number $\tau(f)$ is defined as $\dim_K K[[x]]/(f) + j(f)$ where $\langle f \rangle$ is the ideal generated by $f$. Their results are as follows (see Theorem 5 of [3]):

1. $\mu(f) < \infty$ if and only if $f$ is finitely $\mathcal{R}$–determined.
2. $\tau(f) < \infty$ if and only if $f$ is finitely $\mathcal{K}$–determined.

Since the proofs of Jiang’s theorem and Damon’s result need to use the solution of a differential equation, it seems that their methods do not work in the case of positive characteristic. Motivated by Jiang’s theorem and Damon’s result, following the ideas of [3], we discuss the finite determinacy of singularities $f \in K[[x_1, \ldots, x_n]]$ under the equivalence relation on the power series ring $K[[x]]$ induced by the action of the subgroup of automorphisms preserving the line $x_2 = \cdots = x_n = 0$ or the subgroup of automorphisms preserving a given hypersurface. We try to obtain some results which are similar to Jiang’s theorem, respectively to Damon’s result in case of $X$ is a smooth hypersurface.

In this paper, We have two main results:
(1) For a singularity \( f \in \mathcal{M}^2 \subset K[[x]] \) over an algebraically closed field \( K \) of arbitrary characteristic, the finite \( \mathcal{R}_G \) (or \( \mathcal{K}_G \)--)determinacy of \( f \) is equivalent to the relative \( G \)--isolatedness of the singularity \( f \) (or \( R_f \)), when \( \mathcal{R}_G \) is the subgroup of automorphisms preserving the line \( x_2 = \cdots = x_n = 0 \) and \( \mathcal{K}_G = K[[x]]^* \rtimes \mathcal{R}_G \). (see Theorem 3.7)

(2) Let \( 0 \neq f \in \mathcal{M}^2 \subset K[[x]] \). The finite \( \mathcal{R}_A \) (or \( \mathcal{K}_A \)--)determinacy of \( f \) is equivalent to the relative \( G \)--isolatedness of the singularity \( f \) (or \( R_f \)), when \( \mathcal{R}_A \) is the subgroup of automorphisms preserving a given hypersurface and \( \mathcal{K}_A = K[[x]]^* \rtimes \mathcal{R}_A \). (see Theorem 4.7)

The above results also provide some estimates on the degree of determinacy in positive characteristic (for details, see section 3 and 4).

Moreover, the results we obtain can be applied to classify the \( f \in K[[x]] \) which are finitely \( S \)--determined.

2. Preliminaries

**Lemma 2.1.** (see [7] p. 210) Let \( R \) be any ring and let \( f_1, \ldots, f_n \in \langle x_1, \ldots, x_n \rangle \cdot R[[x_1, \ldots, x_n]] \) be power series. If \( \varphi \) is the endomorphism

\[
\varphi : R[[x_1, \ldots, x_n]] \to R[[x_1, \ldots, x_n]], \quad x_i \mapsto f_i, \quad i = 1, \ldots, n
\]

and the Jacobian matrix \( J(\varphi) \) of \( \varphi \) is the matrix \( ((\varphi_i)_x) \), then \( \varphi \) is an isomorphism if and only if \( \det J(\varphi)(0) \) is a unit in \( K \).

**Lemma 2.2.** (see [3]) Let \( K \) be an algebraically closed field of arbitrary characteristic and \( K[[x]] = K[[x_1, \ldots, x_n]] \). Let \( Q \geq 1 \) be an integer and let \( b_{p,0} \in \mathcal{M}^{Q+p-1} \) and \( b_{p,i} \in \mathcal{M}^{Q+p} \) for \( i = 1, \ldots, n \) and \( p \geq 1 \). Consider the units \( v_p = 1 + b_{p,0} \in K[[x]]^* \) and the automorphisms \( \phi_p \in Aut(K[[x]]) \) given by \( \phi_p : x_i \mapsto x_i + b_{p,i} \) for \( i = 1, \ldots, n \). We denote by

\[
\varphi_p = \phi_p \circ \phi_{p-1} \circ \cdots \circ \phi_1 \in Aut(K[[x]])
\]

the composition of the first \( p \) automorphisms, and we define inductively \( u_p = v_p \cdot \phi_p(u_{p-1}) \), where \( u_0 = 1 \). Then the following hold true:

(a) The sequences \( (\varphi_p(x_i))_{p \geq 1} \) converge in the \( \mathcal{M} \)--adic topology of \( K[[x]] \) to power series \( x_i + b_i \) with \( b_i \in \mathcal{M}^{Q+1} \) for \( i = 1, \ldots, n \). In particular, the map

\[
\varphi : K[[x]] \to K[[x]] : x_i \mapsto x_i + b_i
\]

is a local \( K \)--algebra automorphism of \( K[[x]] \).

(b) The sequence \( (u_p)_{p \geq 1} \) converges in the \( \mathcal{M} \)--adic topology to a unit \( u = 1 + b_0 \in K[[x]]^* \) with \( b_0 \in \mathcal{M}^Q \).
(c) For any power series \( f_0 \in K[[x]] \) the sequence \((\varphi_p(f_0))_{p \geq 1}\) converges in the \( \mathcal{M} \)-adic topology to \( \varphi(f_0) \).

(d) For any power series \( f_0 \in K[[x]] \) the sequence \((u_p \cdot \varphi_p(f_0))_{p \geq 1}\) converges in the \( \mathcal{M} \)-adic topology to \( u \cdot \varphi(f_0) \).

3. Finite \( S \)-determinacy of singularities in positive characteristic, \( S = \mathcal{R}_G, \mathcal{K}_G \)

**Definition 3.1.** Let \( \mathcal{G} \) be the ideal \( \langle x_2, \ldots, x_n \rangle \) of \( K[[x]] \) and \( \mathcal{R} = \text{Aut}(K[[x]]) \). Define \( \mathcal{R}_G = \{ \varphi \in \mathcal{R} | \varphi(\mathcal{G}) = \mathcal{G} \} \). We say that two power series \( f, g \in K[[x]] \) are right line equivalent or \( \mathcal{R}_G \)-equivalent if there is an automorphism \( \varphi \in \mathcal{R}_G \) such that \( f = \varphi(g) \). We denote this relation by \( f \sim_{\mathcal{R}_G} g \). A power series \( f \in K[[x]] \) is called \( k-\mathcal{R}_G \)-determined if for each \( g \in K[[x]] \) such that the same \( k \)-jet as \( f \), \( g \) is right line equivalent to \( f \).

Let \( \mathcal{K}_G = K[[x]]^* \rtimes \mathcal{R}_G \). Two power series \( f, g \in K[[x]] \) are contact line equivalent or \( \mathcal{K}_G \)-equivalent if there is an automorphism \( \varphi \in \mathcal{R}_G \) and a unit \( u \in K[[x]]^* \) such that \( f = u \cdot \varphi(g) \), we denote this relation by \( f \sim_{\mathcal{R}_G} g \). A power series \( f \in K[[x]] \) is \( k-\mathcal{K}_G \)-determined if for each \( g \in K[[x]] \) such that the same \( k \)-jet as \( f \), \( g \) is contact line equivalent to \( f \).

We say that \( f \) is finitely \( \mathcal{R}_G(\mathcal{K}_G) \)-determined if it is \( k-\mathcal{R}_G(\mathcal{K}_G) \)-determined for some positive integer \( k \).

For an \( f \in K[[x]] \), we call the \( K \)-algebra \( R_f = K[[x]]/\langle f \rangle \) the induced hypersurface singularities.

We denote by \( j_G(f) = \mathcal{M} \cdot \langle f_{x_1} \rangle + \mathcal{G} \cdot \langle f_{x_2}, \ldots, f_{x_n} \rangle \) the relative \( \mathcal{G} \)-Jacobian ideal of \( f \), where \( f_{x_i} \) is the formal partial derivative of \( f \) with respect to \( x_i \). We call the associated algebra \( M_G(f) = \frac{K[[x]]}{\mathcal{G}_G(f)} \) the relative \( \mathcal{G} \)-Milnor algebra and its dimension \( \mu_G(f) = \dim_K(\mathcal{M}_G(f)) \) the relative \( \mathcal{G} \)-Milnor number of \( f \). We then call \( f \) a relative \( \mathcal{G} \)-isolated singularity if \( \mu_G(f) < \infty \) or, equivalently, if there is a positive integer such that \( \mathcal{M}^k \subseteq j_G(f) \).

The relative \( \mathcal{G} \)-Tjurina ideal of \( f \) is defined by \( t_G(f) = \langle f \rangle + j_G(f) \). The associated algebra \( T_G(f) = \frac{K[[x]]}{t_G(f)} \) is called the relative \( \mathcal{G} \)-Tjurina algebra of \( f \). The dimension \( \tau_G(f) = \dim_K(T_G(f)) \) of \( T_G(f) \) is called the relative \( \mathcal{G} \)-Tjurina number of \( f \). We then call \( R_f \) a relative \( \mathcal{G} \)-isolated hypersurface singularity if \( \tau_G(f) < \infty \), which is equivalent to the existence of a positive integer \( k \) such that \( \mathcal{M}^k \subseteq t_G(f) \).
Note that the ideal \( j_G(f) \) is basically the tangent space to the orbit of \( f \) under the action of \( R_G \), and similarly that \( tj_G(f) \) is basically the tangent space to the orbit of \( f \) under the action of \( K_G \). The precise statement and its proof will be given in Proposition 3.6.

Let \( f \in K[[x]] \) be a non-zero power series, we denote by \( \text{ord}(f) \) the largest integer \( k \) such that \( f \in M^k \). We set \( \text{ord}(0) = \infty \).

**Theorem 3.2.** Let \( 0 \neq f \in M^2 \) and \( k \in \mathbb{N} \).

(a) If

\[
M^{k+2} \subseteq M^2 \cdot \langle f_{x_1} \rangle + M \cdot G \cdot \langle f_{x_2}, \ldots, f_{x_n} \rangle,
\]

then \( f \) is \((2k - \text{ord}(f) + 2) - R_G\)-determined.

(b) If

\[
M^{k+2} \subseteq M \cdot \langle f \rangle + M^2 \cdot \langle f_{x_1} \rangle + M \cdot G \cdot \langle f_{x_2}, \ldots, f_{x_n} \rangle,
\]

then \( f \) is \((2k - \text{ord}(f) + 2) - K_G\)-determined.

**Proof.** We first prove (b). Let \( o = \text{ord}(f) \). It follows that \( \text{ord}(f_{x_i}) \geq o - 1 \) for all \( i = 1, \ldots, n \) and by assumption we have

\[
M^{k+2} \subseteq M \cdot \langle f \rangle + M^2 \cdot \langle f_{x_1} \rangle + M \cdot G \cdot \langle f_{x_2}, \ldots, f_{x_n} \rangle \subseteq M^{o+1}.
\]

This implies \( k \geq o - 1 \).

Set \( N = 2k - o + 2 \geq k + 1 \), and take \( g \in K[[x]] \) such that \( g - f \in M^{N+1}_+ \), i.e., \( f \) and \( g \) have the same \( N \)-jet. We shall show that \( f \) and \( g \) are \( K_G \)-equivalent, i.e., there exists an automorphism \( \varphi \in R_G \) and a unit \( u \in K[[x]]^* \) such that

\[
g = u \cdot \varphi(f).
\]

We construct \( \varphi \) and \( u \) inductively, i.e., we construct inductively sequences of automorphisms \( (\varphi_p)_{p \geq 1} \) and units \( (u_p)_{p \geq 1} \) such that \( u_p \cdot \varphi_p(f) \) converges in the \( M \)-adic topology to \( u \cdot \varphi(f) \) for some automorphism \( \varphi \in R_G \) and some unit \( u \in K[[x]]^* \) and at the same time

\[
g - u_p \cdot \varphi_p(f) \in M^{N+1+p},
\]

for all \( p \geq 1 \). The latter implies that \( u_p \cdot \varphi_p(f) \) converges to \( g \) as well, and thus

\[
g = u \cdot \varphi(f).
\]
By Lemma 2.2 and its terminology with $Q = N - k \geq 1$ it suffices to construct certain series $b_{p,0} \in \mathcal{M}^{Q^{-p-1}}$, $b_{p,1} \in \mathcal{M}^{Q^p}$, and $b_{p,i} \in \mathcal{M}^{Q^{p+1}} \cdot G \subset \mathcal{M}^{Q^p}$ for $i = 2, \ldots, n$ and $p \geq 1$.

In fact, note that by assumption $g \in \mathcal{M}^{N+1} = \mathcal{M}^{Q^{-1}} \cdot \mathcal{M}^{k+2} \subset \mathcal{M}^Q \cdot t_j G(f)$ there exist $b_{1,0} \in \mathcal{M}^Q$, $b_{1,1} \in \mathcal{M}^{Q+1}$, and $b_{1,i} \in \mathcal{M}^Q \cdot G \subset \mathcal{M}^{Q+1}$ for $i = 2, \ldots, n$ such that

\begin{equation}
(3.1) \quad g - f = b_{1,0} f + b_{1,1} f x_1 + \sum_{i=2}^{n} b_{1,i} f x_i.
\end{equation}

Let $v_1 = 1 + b_{1,0} \in \mathcal{K}[x]^*$ and $\phi_1 : \mathcal{K}[x] \to \mathcal{K}[x] : \ x_i \mapsto x_i + b_{1,i}$, $i = 1, \ldots, n$, where $b_{1,1} \in \mathcal{M}^{Q+1}$, $b_{1,i} \in \mathcal{M}^Q \cdot G \subset \mathcal{M}^{Q+1}$ for $i = 2, \ldots, n$.

Now we prove $\phi_1 \in \mathcal{R}_G$.

In fact, by Lemma 2.1 $\phi_1$ is an automorphism. For any $g \in G = \langle x_2, \ldots, x_n \rangle$, there exist power series $g_2, \ldots, g_n \in \mathcal{K}[x]$ such that $g = g_2 \cdot x_2 + \cdots + g_n \cdot x_n$. We have

\begin{align*}
\phi_1(g) &= \phi_1(g_2) \cdot (x_2 + b_{1,2}) + \cdots + \phi_1(g_n)(x_n + b_{1,n}) \\
&= \sum_{i=2}^{n} \phi_1(g_i) \cdot x_i + \sum_{i=2}^{n} \phi_1(g_i) b_{1,i}.
\end{align*}

Since $b_{1,i} \in \mathcal{M}^Q \cdot G \subset G$, $i = 2, \ldots, n$, we have $\phi_1(g) \in G$.

Next, we want to show that $g - v_1 \cdot \phi_1(f) \in \mathcal{M}^{N+2}$.

If the above formula is true, we can replace $f$ in the above argument by $v_1 \cdot \phi_1(f)$ and go on inductively. Note first that

\begin{align*}
(x_1 + z_1)^{\beta_1} \cdots (x_n + z_n)^{\beta_n} &= \sum_{\gamma_1=0}^{\beta_1} \sum_{\gamma_2=0}^{\beta_2} \cdots \sum_{\gamma_n=0}^{\beta_n} c_{\beta,\gamma} x^{\beta - \gamma} \cdot z^\gamma
\end{align*}

where $c_{\beta,\gamma} = \left( \begin{array}{c} \beta_1 \\ \gamma_1 \end{array} \right) \left( \begin{array}{c} \beta_2 \\ \gamma_2 \end{array} \right) \cdots \left( \begin{array}{c} \beta_n \\ \gamma_n \end{array} \right) \in \mathcal{Z}$. For $f = \sum_{|\beta| \geq 0} k_{\beta} \cdot x^\beta$, consider

\begin{align*}
(3.2) \quad f ((x_1 + z_1), \ldots, (x_n + z_n)) &= \sum_{|\beta| \geq \text{ord}(f)} k_{\beta} \cdot \sum_{\gamma_1=0}^{\beta_1} \sum_{\gamma_2=0}^{\beta_2} \cdots \sum_{\gamma_n=0}^{\beta_n} c_{\beta,\gamma} x^{\beta - \gamma} \cdot z^\gamma \\
&= \sum_{\alpha \in \mathcal{N}^n} w_\alpha \cdot z^\alpha,
\end{align*}
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where

$$ w_\alpha = \sum_{|\beta| \geq \text{ord}(f), \beta \geq \alpha} k_\beta \cdot c_{\beta, \alpha} \cdot x^{\beta - \alpha} $$

if we define $\beta \geq \alpha$ by $\beta_i \geq \alpha_i$ for all $i = 1, 2, \ldots, n$. It follows that

$$ \text{ord}(w_\alpha) = \min \{|\beta| - |\alpha|, |\beta| \geq \text{ord}(f), |\beta| \geq |\alpha|\} \geq o - |\alpha|. $$

We notice that $w_\alpha = \frac{D^\alpha f(x)}{\alpha_1! \alpha_2! \cdots \alpha_n!}$ whenever $\alpha_i < \text{char}(K)$ for all $i = 1, 2, \ldots, n$. In particular, the constant term is $w_0 = f$. For every unit vector $e_i$ $(1 \leq i \leq n)$ $w_{e_i} = f_{x_i}$.

Applying $\phi_1$ to $f$ amounts to substituting $z_1$ by $b_{1,1}$ and $z_i$ by $b_{1,i}$ in (3.2) we thus find $\phi_1(f) = f + f_{x_1} \cdot b_{1,1} + \sum_{i=2}^n f_{x_i} \cdot b_{1,i} + w$, where $w = \sum_{|\alpha| \geq 2} w_{\alpha} \cdot b_{1,1}^{\alpha_1} \cdots b_{1,n}^{\alpha_n}$. Since

$$ \text{ord} \left( w_{\alpha} \cdot b_{1,1}^{\alpha_1} \cdots b_{1,n}^{\alpha_n} \right) \geq \text{ord}(w_\alpha) + \text{ord}(b_{1,1}) \cdot \alpha_1 + \sum_{i=2}^n \text{ord}(b_{1,i}) \cdot \alpha_i $$

$$ \geq o - |\alpha| + (Q + 1) \cdot |\alpha| $$

$$ \geq o + 2 \cdot Q = N + 2, $$

we have $w \in M^{N+2}$. Multiplying $\phi_1(f)$ by $v_1 = 1 + b_{1,0}$ and using (3.1) we get

$$ g - v_1 \cdot \phi_1(f) = g - (1 + b_{1,0}) \cdot (f + f_{x_1} \cdot b_{1,1} + \sum_{i=2}^n f_{x_i} \cdot b_{1,i} + w) $$

$$ = -f_{x_1} \cdot b_{1,1} \cdot b_{1,0} - \sum_{i=2}^n f_{x_i} \cdot b_{1,i} \cdot b_{1,0} - (1 + b_{1,0}) \cdot w. $$

Since

$$ \text{ord}(b_{1,0} \cdot b_{1,i} \cdot f_{x_i}) \geq Q + (Q + 1) + (o - 1) = N + 2, \; i = 1, 2, \ldots, n, $$

we have

(3.3) \quad $g - v_1 \cdot \phi_1(f) \in M^{N+2}$.

Thus we can proceed inductively to construct sequences $\{b_{p,i}\}_{p \geq 1}$ for $i = 0, \ldots, n$ with $b_{p,0} \in M^{Q+p-1}$, $b_{p,1} \in M^{Q+p}$ and $b_{p,i} \in M^{Q+p-1} \cdot G \subseteq M^{Q+p}$ for $i = 2, \ldots, n$. The generalization of (3.3) holds by induction.

Using Lemma 2.2 we have

$$ g - u_p \cdot \varphi_p(f) \in M^{N+1+p}. $$
Again using Lemma 2.2, we obtain an automorphism \((u, \varphi) \in K_G\) such that \(g = u \cdot \varphi(f)\).

The proof for right equivalence can be done in the same lines. The condition \(M_{k+2}^g \subseteq M_k^1 \cdot j_G(f) \subseteq M_{o+1}^g\) implies also that \(k \geq o - 1\). For any \(g\) with

\[
g - f \in \mathcal{M}^{N+1} = \mathcal{M}^{Q-1} \cdot \mathcal{M}^{k+2} \subseteq \mathcal{M}^{Q} \cdot j_G(f)
\]

where \(N = 2k - o + 2 \geq k + 1\) and \(Q = N - k \geq 1\), there exist \(b_{1, i} \in \mathcal{M}^Q\) such that \(i = 2, \ldots, n\) with

\[
g - f = b_{1, i} \cdot f_x + b_{1, 2} \cdot f_x^2 + \cdots + b_{1, n} \cdot f_x^n.
\]

We can then define \(\phi_l\) as above. It is easy to show

\[
g - \phi_l(f) = h \in \mathcal{M}^{N+2}.
\]

Going on by induction and applying Lemma 2.2, we get an automorphism \(\varphi \in R_G\) such that \(g = \varphi(f)\).

\[\Box\]

**Corollary 3.3.** Let \(0 \neq f \in \mathcal{M}^2 \subseteq K[[x]]\).

(a) If \(\mu_G(f) < \infty\), then \(f\) is \((2\mu_G(f) - \text{ord}(f)) - \mathcal{R}_G\)-determined.

(b) If \(\tau_G(f) < \infty\), then \(f\) is \((2\tau_G(f) - \text{ord}(f)) - K_G\)-determined.

The converse also holds in arbitrary characteristic.

**Theorem 3.4.** Let \(0 \neq f \in \mathcal{M} \subseteq K[[x]]\).

(a) If \(f\) is \(R_G - k\)-determined, then \(M_{k+1} \subseteq j_G(f)\).

(b) If \(f\) is \(K_G - k\)-determined, then \(M_{k+1} \subseteq (f) + j_G(f)\).

The proof of Theorem 3.4 is analogous to the result established in [3]. Before we begin the proof, we need some notations.

Denote \(J_l = K[[x]]/M_l^{l+1}\) the space of \(l\)-jets of power series in \(K[[x]]\). Each \(K\)-algebra automorphism \(\varphi\) of \(K[[x]]\) is a tuple \((\varphi_1, \varphi_2, \ldots, \varphi_n) \in K[[x]]^n\) of power series such that \(\varphi_i(0) = 0\), for all \(i = 1, 2, \ldots, n\) and \(\text{Det} \left( \frac{\partial \varphi_i}{\partial x_j} (0) \right)_{i,j=1,2,\ldots,n}\) is invertible. The \(l\)-jet of the automorphism \(\varphi\) is \(\text{jet}_l(\varphi) = (\text{jet}_l(\varphi_1), \ldots, \text{jet}_l(\varphi_n))\). The \(l\)-jet of the right line equivalence group is \(R_{G,l} = \{ \text{jet}_l(\varphi) | \varphi \in \mathcal{R}_G \}\) and the \(l\)-jet of the contact line equivalence group is \(K_{G,l} = \text{jet}_l(K[[x]]^n) \times \mathcal{R}_{G,l}\). \(K_{G,l}\) acts on \(J_l\) via

\[\phi_l : K_{G,l} \times J_l \rightarrow J_l : (\text{jet}_l(u), \text{jet}_l(\varphi), \text{jet}_l(f)) \mapsto \text{jet}_l(u \cdot \varphi(f))\,.
\]

Similarly, we define the action of the \(l\)-jet \(\mathcal{R}_{G,l}\) on \(J_l\).
Remark 3.5. (a) From [3], we know that \(J_l\) is an affine space and \(K_l\) and \(R_l\) are affine algebraic groups acting on \(J_l\) via a regular separable algebraic action.

(b) \(K_{G,l}\) and \(R_{G,l}\) are affine algebraic groups acting on \(J_l\) via a regular separable algebraic action.

In fact, given \(\varphi = (\varphi_1, \varphi_2, \ldots, \varphi_n) \in R_l\), we have \(\varphi_i = \varphi(x_i) \in G\) for \(i = 2, \ldots, n\). It implies that \(\frac{\partial \varphi_i}{\partial x_1}(x_1, 0, \ldots, 0) = 0\) for \(i = 2, \ldots, n\).

Let \(\text{jet}_l(f) = \sum_{|\alpha| = 0}^l a_\alpha x_2^{\alpha_2} \cdots x_n^{\alpha_n}\), \(\text{jet}_l(\varphi_i) = \sum_{|\beta| = 1}^l b_i,\beta x_1^{\beta_1}x_2^{\beta_2} \cdots x_n^{\beta_n}\) and \(\text{jet}_l(u) = \sum_{|\gamma| = 0}^l c_\gamma x_1^{\gamma_1}x_2^{\gamma_2} \cdots x_n^{\gamma_n}\).

We can choose coordinate variables \((a_\alpha, b_i, c_\gamma)_{\alpha, i, \gamma}\) on \(K_l \times J_l\) with \(c_0 \neq 0\) and \(\text{Det}(B) \neq 0\) where \(B = (B_{ij})\) with \(B_{ij} = \frac{\partial \varphi_i}{\partial x_j}(0) = b_i, e_j\) and \(e_j\) the \(j\)-th canonical basic vectors in \(\mathbb{Z}^n\).

We note that \(K_{G,l} \times J_l\) is a subvariety of \(K_l \times J_l\). This is because \(K_{G,l} \times J_l\) is defined by a system of equations \(b_i, k, e_1 = 0\), for all \(i = 2, \ldots, n\) and \(k = 1, \ldots, l\). Again by Remark 2 of [3], the extension \(K(K_l \times J_l)\) of the field \(K\) is a purely transcendental extension of \(K(J_l)\) and it is thus a separably generated extension. Since \(K_{G,l} \times J_l \subseteq K_l \times J_l\), we have \(K(K_{G,l} \times J_l)\) is a separably generated extension of \(K(J_l)\).

Now we can obtain the tangent space to the orbits also in positive characteristic.

Proposition 3.6. Let \(f \in K[[x]]\). Then the tangent space to the orbit of \(\text{jet}_l(f)\) under the action of \(R_{G,l}\) and \(K_{G,l}\) considered as a subspace of \(J_l\) are

\[
T_{\text{jet}_l(f)}(R_{G,l} \cdot \text{jet}_l(f)) = \left(\text{jet}_l(f) + M^{l+1}\right) / M^{l+1}
\]

\[
T_{\text{jet}_l(f)}(K_{G,l} \cdot \text{jet}_l(f)) = \left(\langle f \rangle + \text{jet}_l(f) + M^{l+1}\right) / M^{l+1}
\]

Proof. Let \(G\) be one of the two above groups, then the action of \(G\) on \(J_l\) induces a surjective separable morphism \(G \rightarrow G \cdot \text{jet}_l(f)\) of smooth varieties. As \(K(K_{G,l} \times J_l)\) is a separably generated extension of \(K(J_l)\), the induced differential map on the tangent spaces is generically surjective (see e.g. the proof of [8], Ch.3. Lemma 10.5,]).

Because each point in \(G\) can be translated to the identity element of \(G\) and this translation is an isomorphism, it thus suffices to understand the image of the tangent space to \(G\) at the identity element of \(G\) and its image under the differential map. We restrict here to the case \(G = K_{G,l}\) since the proof for \(R_{G,l}\) is analogous to \(K_{G,l}\).
We now describe the tangent space to $K_{G,l}$ at $(1, id)$, through the local $K$–algebra homomorphisms from the local ring of $K_{G,l}$ to $K[[t]]$ with $t^2 = 0$. In this sense, a tangent vector of $K_{G,l}$ at $(1, id)$ can be represented by the residue class modulo $M_{l+1}$ of a tuple $(1+t\cdot a, id+t\cdot \phi)$ in $K_{G,l}$ with $a \in K[[x]]$ and $\phi = (\phi_1, \phi_2, \ldots, \phi_n)$, where $\phi_1 \in M$ and $\phi_i \in G$, $i = 2, \ldots, n$.

The tangent space to $K_{G,l} \cdot \text{jet}_l(f)$ at $\text{jet}_l(f)$ can be described as follows. We apply the differential map by acting with the above tuple on $f$ modulo $M_{l+1}$: Expanding the power series as in (3.2), we have

$$(1 + t \cdot a) \cdot f ((x) + t\phi) = f + t \cdot \left( a \cdot f + f_{x_1} \phi_1 + \sum_{i=2}^{n} f_{x_i} \phi_i \right) + t^2 h(x, t).$$

Hence, in $K[[x]][[t]]/(t^2)$,

$$(1 + t \cdot a) \cdot f ((x) + t \cdot \phi) = f + t \cdot \left( a \cdot f + f_{x_1} \cdot \phi_1 + \sum_{i=2}^{n} f_{x_i} \cdot \phi_i \right).$$

In $J_l$ this tangent vector is just the $l$–jet of

$$a \cdot f + f_{x_1} \cdot \phi_1 + \sum_{i=2}^{n} f_{x_i} \cdot \phi_i.$$

This implies that

$$T_{\text{jet}_l(f)} (K_{G,l} \cdot \text{jet}_l(f)) = \left( \langle f \rangle + j_G(f) + M_{l+1} \right) / M_{l+1}. \tag*{$\square$}$$

Now we prove Theorem 3.4.

**Proof.** We give only the proof of $K_{G,l}$–determinacy since the proof of the other case is analogous. If $f$ is $k-K_{G,l}$–determined and $g \in M_{k+1}$, then for any $t \in K$ the $(k+1)$–jet $\text{jet}_{k+1}(f) + t \cdot \text{jet}_{k+1}(g)$ is in the orbit of $\text{jet}_{k+1}(f)$ under $K_{G,k+1}$. Hence $\text{jet}_{k+1}(g) \in T_{\text{jet}_l(f)} (K_{G,k+1} \cdot \text{jet}_{k+1}(f)) = \left( \langle f \rangle + j_G(f) + M_{k+2} \right) / M_{k+2}$. This implies that

$$g \in \langle f \rangle + j_G(f) + M_{k+2},$$

and hence

$$M_{k+1} \subseteq \langle f \rangle + j_G(f) + M_{k+2}. $$
By Nakayama’s Lemma we get $\mathcal{M}^{k+1} \subseteq \langle f \rangle + j_G(f)$.

From the formulas in Proposition 3.6, the geometrical meaning of the ideals $j_G(f)$ and $tj_G(f)$ are the tangent space to the orbit of $f$ under the action of $\mathcal{R}_G$ and $\mathcal{K}_G$ respectively.

Combining Corollary 3.3 and Theorem 3.4, we obtain:

**Theorem 3.7.** Let $0 \neq f \in \mathcal{M}^2 \subset K[[x]]$ be a power series.

1. $f$ is a relative $\mathcal{G}$—isolated singularity if and only if $f$ is finitely $\mathcal{R}_G$—determined.

2. $\mathcal{R}_f$ is a relative $\mathcal{G}$—isolated hypersurface singularity if and only if $f$ is finitely $\mathcal{K}_G$—determined.

4. **finite $S$—determinacy of singularities in positive characteristic, $S = \mathcal{R}_G$, $\mathcal{K}_G$**

**Definition 4.1.** Let $h \in K[[x]]$ with $h(0) = 0$ and $\frac{d h}{d x_n}(0) \neq 0$. For a hypersurface ideal $\mathcal{A} = \langle h \rangle$ of $K[[x]]$, $\mathcal{R}_\mathcal{A} = \{ \varphi \in \mathcal{R} \mid \varphi(\mathcal{A}) = \mathcal{A} \}$.

Two power series $f, g \in K[[x]]$ are right hypersurface equivalent or $\mathcal{R}_\mathcal{A}$—equivalent if there is an automorphism $\varphi \in \mathcal{R}_\mathcal{A}$ such that $f = \varphi(g)$. We denote this relation by $f \sim_{\mathcal{R}_\mathcal{A}} g$.

A power series $f \in K[[x]]$ is $k$—$\mathcal{R}_\mathcal{A}$—determined if for each $g \in K[[x]]$ such that the same $k$—jet as $f$, $g$ is right hypersurface equivalent to $f$.

We define $\mathcal{K}_\mathcal{A} = K[[x]]^* \rtimes \mathcal{R}_\mathcal{A}$. Two power series $f, g \in K[[x]]$ are contact hypersurface equivalent or $\mathcal{K}_\mathcal{A}$—equivalent if there is an automorphism $\varphi \in \mathcal{R}_\mathcal{A}$ and a unit $u \in K[[x]]^*$ such that $f = u \cdot \varphi(g)$, where $(u, \varphi) \in K$. We denote this relation by $f \sim_{\mathcal{K}_\mathcal{A}} g$.

A power series $f \in K[[x]]$ is $k$—$\mathcal{K}_\mathcal{A}$—determined if for each $g \in K[[x]]$ such that the same $k$—jet as $f$, $g$ is contact hypersurface equivalent to $f$.

We say that $f$ is finitely $\mathcal{R}_\mathcal{A}(\mathcal{K}_\mathcal{A})$—determined if it is $k$—$\mathcal{R}_\mathcal{A}(\mathcal{K}_\mathcal{A})$—determined for some positive integer $k$.

For a power series $f \in K[[x]]$, Let

$$j_\mathcal{A}(f) = \mathcal{M} \cdot (h_{x_n} \cdot f_{x_i} - h_{x_i} \cdot f_{x_n} \mid i = 1, \ldots, n - 1) + \mathcal{A} \cdot \langle f_{x_n} \rangle$$

be the relative $\mathcal{A}$—Jacobian ideal of $f$.

The relative $\mathcal{A}$—Milnor algebra $M_\mathcal{A}(f)$ of $f$ is defined as $M_\mathcal{A}(f) = K[[x]] / j_\mathcal{A}(f)$. We call its dimension $\mu_\mathcal{A}(f) = \dim_K(M_\mathcal{A}(f))$ the relative $\mathcal{A}$—Milnor number of $f$. We call $f$ a relative $\mathcal{A}$—isolated singularity if $\mu_\mathcal{A}(f) < \infty$ or, equivalently, if there is a positive integer such that $\mathcal{M}^k \subseteq j_\mathcal{A}(f)$. 

In the complex case, when $2.1 \varphi$ let $\varphi$.

J.W.Bruce defined the Milnor number of $f$ and the associated relative $\varphi$-Tjurina algebra of $f$ is $T_\varphi(f) = \frac{K[[x]]}{t_{fA}(f)}$.

The dimension $\tau_\varphi(f) = \dim_K(T_\varphi(f))$ of $T_\varphi(f)$ is called the relative $\varphi$-Tjurina number of $f$. We then call $R_f$ a relative $\varphi$-isolated hypersurface singularity if $\tau_\varphi(f) < \infty$, which is equivalent to the existence of a positive integer $k$ such that $M^k \subseteq t_{fA}(f)$.

Note that the ideal $j_\varphi(f)$ is basically the tangent space to the orbit of $f$ under the action of $R_\varphi$, and similarly that $t_{fA}(f)$ is basically the tangent space to the orbit of $f$ under the action of $K_\varphi$. The precise statement and its proof will be given in Proposition 4.4.

**Remark 4.2.** In the complex case, when $(X, 0)$ is the germ of an analytic subvariety of $(\mathbb{C}^n, 0)$ and $f$ again a function germ on $\mathbb{C}^n$ at 0, J.W.Bruce defined the Milnor number of $f$ on $X$ by

$$\mu_X(f) = \dim_{\mathbb{C}}O_{X, 0}/j_X(f)$$

(see [4]). If $X$ is a hypersurface defined by $h : \mathbb{C}^n \to \mathbb{C}$ in analytic space $(\mathbb{C}^n, 0)$, where $h(0) = 0$ and $h_{x_n}(0) \neq 0$, then

$$\Theta_{X, 0} = \left\langle h_{x_n} \cdot \frac{\partial}{\partial x_i} - h_{x_i} \cdot \frac{\partial}{\partial x_n} \mid i = 1, \ldots, n - 1 \right\rangle + \left\langle h \cdot \frac{\partial}{\partial x_n} \right\rangle$$

and

$$j_X(f) = \left\langle h_{x_n} \cdot f_{x_i} - h_{x_i} \cdot f_{x_n} \mid i = 1, \ldots, n - 1 \right\rangle + \left\langle h \cdot f_{x_n} \right\rangle.$$  

However, the number $\mu_X(f)$ does not coincide with the number $\mu_\varphi(f)$.

The number $\mu_X(f)$ coincides with the usual Milnor number $\mu(f)$ in the case that $X = \emptyset$. On the other hand, it is not the codimension of the orbit of $f$ under the group action of $R_X$, while this is the case for the number $\mu_\varphi(f)$ under the group action of $R_\varphi$.

**Theorem 4.3.** Let $0 \neq f \in M \subseteq K[[x]]$.

(a) If $f$ is $R_\varphi - k$-determined, then $M^{k+1} \subseteq j_\varphi(f)$.

(b) If $f$ is $K_\varphi - k$-determined, then $M^{k+1} \subseteq t_{fA}(f)$.

In order to prove Theorem 4.3, we need some facts and propositions. Consider the map $\psi : K[[x]] \to K[[x]], x_i \mapsto x_i, (1 \leq i \leq n - 1), x_n \mapsto h$. By Lemma 2.1, $\psi$ is an isomorphism. Let $\varphi$ be an element of $R_\varphi$.

Set $\varphi = \psi^{-1} \circ \varphi \circ \psi$. Then $\varphi = \psi \circ \varphi \circ \psi^{-1}$. We have

$$\varphi(\langle h \rangle) = \langle h \rangle \Leftrightarrow \varphi(\langle x_n \rangle) = \langle x_n \rangle.$$  

So $\varphi = \{ \varphi \in R \mid \varphi(A) = A \}$ is isomorphic to $\varphi = \{ \varphi \in R \mid \varphi(\langle x_n \rangle) = \langle x_n \rangle \}$.
The finite $S$-determinacy of singularities

The $l$-jet of $\mathcal{R}_A$ is $\mathcal{R}_{A,l} = \{ \text{jet}_l(\varphi) \mid \varphi \in \mathcal{R}_A \}$ and the $l$-jet of $\mathcal{K}_A$ is $\mathcal{K}_{A,l} = \text{jet}_l(\mathcal{K}[[x_1, \ldots, x_n]]) \ltimes \mathcal{R}_{A,l}$.

Now we show that $\mathcal{K}_{A,l}$ and $\mathcal{R}_{A,l}$ are affine algebraic groups acting on $J_l$ via a regular separable algebraic action.

For $u \in \mathcal{K}[[x]]^*$, $f \in \mathcal{K}[[x]]$, let $\text{jet}_l(u) = \sum_{|\gamma|=0}^l c_\gamma x^\gamma$, $\text{jet}(f) = \sum_{|\alpha|=0}^l a_\alpha x^\alpha$. If $\varphi \in \mathcal{R}_A$, then $\varphi = (\varphi_1, \ldots, \varphi_n)$ and there exists a $g \in \mathcal{K}[[x]]$ such that $\varphi(x_n) = x_n \cdot g$. Let $\text{jet}_l(\varphi_i) = \sum_{|\beta|=1}^l b_{i,\beta} x^\beta$, and $\text{jet}(g) = \sum_{|\lambda|=0}^l g_\lambda x^\lambda$. Then $\text{jet}_l(\varphi_n) = \text{jet}_l(x_n \cdot g)$. We can obtain a system of equations by comparing the coefficients of the monomials $x^\beta$ on both sides of the equation $\text{jet}_l(\varphi_n) = \text{jet}_l(x_n \cdot g)$. So the coordinates $b_{n,\beta}$ are given by polynomial maps $b_{n,\beta} = W_\beta(g_\lambda)$, where $0 \leq |\lambda| \leq l-1, 1 \leq |\beta| \leq l$, and $g_0 \neq 0$. In fact, if $g_0 = 0$, then the first term of $\varphi_n$ is $b_{n,\beta} x_1^{\beta_1} x_2^{\beta_2} \cdots x_n^{\beta_n}$, where $|\beta| = 2$, so that $(\varphi_n)(x_i)(0) = 0$, $i = 1, \ldots, n$. It is a contradiction to the fact that $\det J(\varphi)(0)$ is a unit in $\mathcal{K}$.

So we can take coordinates

$$(a_\alpha, b_{i,\beta}, g_\lambda, c_\gamma)_\alpha, i, \beta, \gamma, \lambda, \quad 1 \leq i < n,$$

$$0 \leq |\beta| \leq l, 1 \leq |\alpha|, |\gamma| \leq l, 0 \leq |\lambda| \leq l-1$$

on $\mathcal{K}_{A,l} \times J_l$, it satisfies the following conditions: (1) $c_0 \neq 0$; (2) $\det(B) \neq 0$ where $B = (B_{ij})$ with $B_{ij} = (\varphi_i)_x(0) = b_{i,e_j}$ where $e_j$ is the $j$-th canonical basis vector in $\mathbb{Z}^n$ and the coordinates $b_{n,e_j} = W_{e_j}(g_\lambda)$, $0 \leq |\lambda| \leq l-1, 1 \leq j \leq n$; (3) $g_0 \neq 0$. Using in the same manner the coordinates $(a'_\delta)_{|\delta|=0, \ldots, l}$ on the target space, we define the action by polynomial maps

$$a'_\delta = F_\delta(a_\alpha, b_{i,\beta}, g_\lambda, c_\gamma).$$

It is important to note that the inverse of this action is given by the rational maps

$$a_\alpha = \frac{G_\delta(a'_\delta, b_{i,\beta}, g_\lambda, c_\gamma)}{H_\delta(a'_\delta, b_{i,\beta}, g_\lambda, c_\gamma)}.$$

The reason for this is that we can solve the $a_\alpha$ step by step starting with Cramer’s rule. This property ensures the extension of the field of
rational functions induced by the action of $\Phi_t$. We have

$$K(J_t) = K(a'_d) \subset K(K_{\mathcal{A}, I} \times J_t) = K(a_\alpha, b_\beta, g_\lambda, c_\gamma)$$

$$= K(a'_d, b_\beta, g_\lambda, c_\gamma) = K(J_t)(b_\beta, g_\lambda, c_\gamma).$$

The $b_\beta, g_\lambda$ and $c_\gamma$ are algebraically independent over $K(a_\alpha)$. Comparing transcendence degrees they must be also algebraically independent over $K(J_t)$. Thus $K(K_{\mathcal{A}, I} \times J_t)$ is a purely transcendental extension of $K(J_t)$, and it is a separably generated extension in the sense of [8, p.27]. Hence $K_{\mathcal{A}, I}$ operates separably on $J_t$.

Let $F : \mathcal{R}_A \to \{ \varphi : \varphi \in \mathcal{R} \text{ and } \varphi(\langle x_n \rangle) = \langle x_n \rangle \} = \varphi \mapsto \psi^{-1} \circ \varphi \circ \psi$. Then $F$ from $\mathcal{R}_A$ to $\mathcal{R}_{\mathcal{A}, I}$ is one-to-one and onto. So $K(K_{\mathcal{A}, I} \times J_t)$ is a separably generated extension of $K(J_t)$.

Now we can prove the following proposition.

**Proposition 4.4.** Let $f \in K[[x]]$. The tangent space to the orbit of $\text{jet}_t(f)$ under the actions of $\mathcal{R}_{\mathcal{A}, I}$ and $K_{\mathcal{A}, I}$ considered as subspaces of $J_t$ are, respectively,

$$T_{\text{jet}_t(f)}(\mathcal{R}_{\mathcal{A}, I} \cdot \text{jet}_t(f)) = \left(j_A(f) + \mathcal{M}^{l+1}\right)/\mathcal{M}^{l+1}$$

and

$$T_{\text{jet}_t(f)}(K_{\mathcal{A}, I} \cdot \text{jet}_t(f)) = \left(tj_A(f) + \mathcal{M}^{l+1}\right)/\mathcal{M}^{l+1}.$$ 

**Proof.** We note that the action of $G = \mathcal{R}_{\mathcal{A}, I}$ or $G = K_{\mathcal{A}, I}$ on $J_t$ induces a surjective separable morphism $G \to G \cdot \text{jet}_t(f)$ of smooth varieties. The proof is similar to the first part of the proof of Proposition 3.6.

We give only the proof in the case $G = K_{\mathcal{A}, I}$ since the proof of $\mathcal{R}_{\mathcal{A}, I}$ is completely similar to the case of $K_{\mathcal{A}, I}$.

Now we compute the tangent space $T_{\text{jet}_t(f)}(K_{\mathcal{A}, I} \cdot \text{jet}_t(f))$.

Let $\psi$ be the map $\psi : K[[x]] \to K[[x]], x_i \mapsto x_i, (1 \leq i \leq n-1), x_n \mapsto h$. By Lemma 2.1, $\psi$ is an isomorphism. The tangent space to $K_{\mathcal{A}, I}$ at $(1, id)$ can be described via the local $K$-algebra homomorphisms from the local ring of $K_{\mathcal{A}, I}$ at $(1, id)$ to $K[[t]]/(t^2)$. A tangent vector of $K_{\mathcal{A}, I}$ at $(1, id)$ is represented by the residue class modulo $\mathcal{M}^{l+1}$ of a tuple $(1 + t \cdot a, id + t \cdot \varphi^*)$ with $a \in K[[x]]$ and $\varphi^* = (\varphi_1^*, \varphi_2^*, \ldots, \varphi_n^*)$ where $\varphi_i^* \in \mathcal{M}$, $i = 1, \ldots, n$. This means in particular that $t \in K[[t]]/(t^2)$, i.e.,

$\varphi_i^* = (1 + t \cdot a, id + t \cdot \varphi^*)$ with $a \in K[[x]]$ and $\varphi^* = (\varphi_1^*, \varphi_2^*, \ldots, \varphi_n^*)$.
If \((1 + t \cdot a, \ id + t \cdot \varphi^*)\) is a tangent vector of \(K_{A,l}\) at \((1, \ id)\), then

\[
\delta = \sum_{i=1}^{n} \varphi_i^* \frac{\partial}{\partial x_i}
\]

is a derivation that satisfies \(\delta(h) \subseteq \langle h \rangle\). Thus there exists a power series \(g \in K[[x_1, \ldots, x_n]]\) such that

\[
g \cdot h = \delta(h) = \sum_{i=1}^{n} \varphi_i^* \frac{\partial h}{\partial x_i}.
\]

This implies that

\[
\varphi_n^* = \frac{1}{x_n} \cdot \left( g \cdot h - \sum_{i=1}^{n-1} \varphi_i^* \cdot hx_i \right).
\]

Plugging this into the definition of \(\delta\) we get

\[
(4.1) \quad \delta = \frac{1}{x_n} \cdot \left( \sum_{i=1}^{n-1} \varphi_i^* \cdot (hx_n \cdot \partial h/hx_i) - h_{x_i} \cdot \partial h/hx_n + g \cdot \partial h/hx_n \right).
\]

Applying this to \(f\) we find that

\[
\delta(f) \in j_{A}(f),
\]

since \(\varphi_i^* \in M\) for \(i = 1, \ldots n - 1\) and \(g \cdot h \in A\). Then we have

\[
(1 + ta) \cdot f(x + t\varphi^*) = f + t \cdot (af + \delta(f))
\]

and

\[
a f + \delta(f) \in t_{j_{A}}(f).
\]

Thus \((4.1)\) implies that:

\[
T_{j_{t_{l}}} (K_{A,l} \cdot j_{t_{l}}(f)) = \left( t_{j_{A}}(f) + M^{l+1} \right) / M^{l+1}.
\]

Now we prove Theorem 4.3.

**Proof.** We only prove the \(K_{A,k+1}\)-determinacy since the other case is completely analogous. If \(f\) is \(k\)-determined and \(g \in M^{k+1}\), then for any \(t \in K\) the \((k + 1)\)-jet \(\text{jet}_{k+1}(f) + t \cdot \text{jet}_{k+1}(g)\) is in the orbit of \(\text{jet}_{k+1}(f)\) under \(K_{A,k+1}\). So

\[
\text{jet}_{k+1}(g) \in T_{j_{t_{k+1}}} (K_{A,k+1} \cdot j_{t_{k+1}}(f)) = \left( t_{j_{A}}(f) + M^{k+2} \right) / M^{k+2}.
\]
This implies that $g \in t j_A(f) + M^{k+2}$, and hence $M^{k+1} \subseteq t j_A(f) + M^{k+2}$.

By Nakayama’s Lemma we get $M^{k+1} \subseteq t j_A(f)$. □

**Theorem 4.5.** Let $0 \neq f \in M^2$ and $k \in \mathbb{N}$
(a) If $M^{k+2} \subseteq M \cdot j_A(f)$, then $f$ is $(2k - \text{ord}(f) + 2) - R_A -$ determined.
(b) If $M^{k+2} \subseteq M \cdot t j_A(f)$, then $f$ is $(2k - \text{ord}(f) + 2) - K_A -$ determined.

**Proof.** We first prove (b). Let $o = \text{ord}(f)$. By assumption and the fact that $\text{ord}(f_{x_i}) \geq o - 1$ for $i = 1, \ldots, n$, we have $M^{k+2} \subseteq M \cdot t j_A(f) \subseteq M^{o+1}$. This implies that $k \geq o - 1$.

Set $N = 2k - o + 2 \geq k + 1$, and take a $g \in K[[x]]$ such that $g - f \in M^{N+1}$, i.e., $f$ and $g$ have the same N-jet. The key point of the proof is to show that $f$ and $g$ are contact hypersurface equivalent, i.e., there are an automorphism $\varphi \in R_A$ and a unit $u \in K[[x]]^*$ such that $g = u \cdot \varphi(f)$.

In order to construct $\varphi$ and $u$, we must use Lemma 2.2 and consider the following three cases:

(1): $h = x_n K[[x_1, \ldots, x_{n-1}]]$;
(2): $h = x_n + h_1(x_1, \ldots, x_{n-1})$;
(3): $h = H_1(x_1, \ldots, x_n) \cdot x_n + h_1(x_1, \ldots, x_{n-1})$, where $H_1 \in K[[x]]$.

Case (1): Let $h \in x_n K[[x_1, \ldots, x_{n-1}]]$. Then there exits $H \in K[[x]]$
such that $h = H(x) \cdot x_n$.

Set $Q = N - k \geq 1$, by assumption

$$g - f \in M^{N+1} = M^{Q-1} \cdot M^{k+2} \subseteq M^{Q} \cdot \langle f \rangle + M^{Q} \cdot j_A(f)$$

$$= M^{Q} \cdot \langle f \rangle + M^{Q} \cdot A \cdot \langle f_{x_n} \rangle$$

$$+ M^{Q+1} \cdot \langle \{ h_{x_n} \cdot f_{x_j} - h_{x_j} \cdot f_{x_n} : 1 \leq j < n \} \rangle.$$

Thus there exist $a_{1,0} \in M^Q$, $a_{1,j} \in M^{Q+1}$, $1 \leq j < n$ and $a_{1,n} \in M^{Q} \cdot A \subseteq M^{Q+1}$ such that

$$g - f = a_{1,0} f + \sum_{1 \leq j < n} a_{1,j} (h_{x_n} \cdot f_{x_j} - h_{x_j} \cdot f_{x_n}) + a_{1,n} f_{x_n}$$

(4.2) $$= a_{1,0} f + \sum_{j=1}^{n-1} (a_{1,j} h_{x_n}) f_{x_j} - \sum_{j=1}^{n-1} (a_{1,j} h_{x_j}) f_{x_n} + a_{1,n} f_{x_n}.$$ Let $b_{1,0} = a_{1,0}$, $b_{1,j} = a_{1,j} h_{x_n}$, $j = 1, \ldots, n-1$, $b_{1,n} = -\sum_{j=1}^{n-1} (a_{1,j} h_{x_j}) + a_{1,n}$, then

$$g - f = b_{1,0} \cdot f + \sum_{j=1}^{n} b_{1,j} \cdot f_{x_j}.$$
Now define $v_1 = 1 + b_{1,0} \in K[[x]]^*$ and $\phi_1 : K[[x]] \to K[[x]] : x_j \mapsto x_j + b_{1,j} = x_j + a_{1,j}h_{x_n}, (j = 1, \ldots, n - 1), x_n \mapsto x_n + b_{1,n} = x_n - \sum_{j=1}^{n-1} (a_{1,j}h_{x_j}) + a_{1,n}$. We want to show that
\begin{equation}
    g - v_1 \cdot \phi_1(f) \in M^{N+2}.
\end{equation}
If the formula (4.3) is true, we can replace $f$ in the above argument by $v_1 \cdot \phi_1(f)$ and go on inductively.

For $f = \sum_{|\beta| \geq 0} k_\beta \cdot x^\beta$, we have (3.2). Applying $\phi_1$ to $f$ amounts to substituting $z_j$ by $a_{1,j} \frac{\partial}{\partial z_j}$, $j = 1, \ldots, n-1$, and $z_n$ by $\left(-\sum_{j=1}^{n-1} a_{1,j}h_{x_j}\right) + a_{1,n}$ in (3.2). Thus we find that
\[
    \phi_1(f) = f + \sum_{i=1}^{n-1} f_{x_i} \cdot (a_{1,i}h_{x_n}) + f_{x_n} \cdot \left(-\sum_{j=1}^{n-1} a_{1,j}h_{x_j} + a_{1,n}\right) + r
\]
where
\[
    r = \sum_{|\alpha| \geq 2} w_\alpha \cdot (a_{1,1}h_{x_n})^{\alpha_1}(a_{1,2}h_{x_n})^{\alpha_2} \cdots (-\sum_{j=1}^{n-1} a_{1,j}h_{x_j} + a_{1,n})^{\alpha_n}.
\]
Since $h_{x_n}(0) \neq 0$ we obtain
\[
    \ord(r) \geq \ord(w_\alpha) + \sum_{i=1}^{n-1} \ord(a_{1,i}h_{x_n}) \cdot \alpha_i + \ord(-\sum_{j=1}^{n-1} a_{1,j}h_{x_j} + a_{1,n}) \cdot \alpha_n \geq o+|\alpha|+(Q+1)\cdot|\alpha| \geq o+2\cdot Q = N+2, \quad r \in M^{N+2}.
\]
Multiplying $\phi_1(f)$ by $v_1 = 1 + a_{1,0}$ and using (4.2) we get $g - v_1 \cdot \phi_1(f) = -\left(\sum_{i=1}^{n-1} f_{x_i} \cdot (a_{1,i}h_{x_n}) + f_{x_n} \cdot (-\sum_{j=1}^{n-1} a_{1,j}h_{x_j} + a_{1,n})\right) \cdot a_{1,0} - (1 + a_{1,0})r$.

Since $\ord \left[ a_{1,0} \cdot (a_{1,i}h_{x_n}) \cdot f_{x_i} \right] \geq Q + (Q+1) + (o-1) = N + 2$ and $\ord \left[ a_{1,0} \cdot (-\sum_{j=1}^{n-1} a_{1,j}h_{x_j} + a_{1,n}) \cdot f_{x_n} \right] \geq Q + (Q+1) + (o-1) = N + 2$, we have $g - v_1 \cdot \phi_1(f) \in M^{N+2}$. This proves (4.3).

Now, we prove $\phi_1(\langle h \rangle) = \langle h \rangle$.

We take a map $\psi : K[[x]] \to K[[x]], \quad x_i \mapsto x_i, (1 \leq i \leq n - 1), x_n \mapsto h$. By Lemma 2.2, $\psi$ is an isomorphism and $\psi$ is the identity
on $K[[x_1, \ldots, x_{n-1}]]$. Because $K[[x_1, \ldots, x_n]] = K[[x_1, \ldots, x_{n-1}]][[x_n]]$ and the elements of $K[[x_1, \ldots, x_n]]$ which are not in $\langle x_n \rangle$ are those with nonzero term in $K[[x_1, \ldots, x_{n-1}]]$, $\psi$ preserves this subset. Since $\psi$ is an isomorphism, it follows that $\psi(\langle x_n \rangle) = \langle x_n \rangle$. In particular, the image $\psi(x_n) = h$ of the generator $x_n$ of $\langle x_n \rangle$ is a generator of $\langle x_n \rangle$. We have $\langle x_n \rangle = \langle h \rangle$.

For any $g = g_n(x_1, \ldots, x_n)x_n \in \langle x_n \rangle$,

\[
\phi_1(g) = \phi_1(g_n) \cdot \phi_1(x_n) = \phi_1(g_n) \cdot \left(x_n - \sum_{j=1}^{n-1} (a_{1,j}h_{x_j}) + a_{1,n}\right)
\]

\[
= \phi_1(g_n)x_n - \phi_1(g_n) \cdot \left(\sum_{j=1}^{n-1} a_{1,j}h_{x_j}\right) + \phi_1(g_n) \cdot a_{1,n}.
\]

From the fact that $h_{x_j} = (H(x_1, \ldots, x_n) \cdot x_n)_{x_j} = H_{x_j} \cdot x_n$, $j = 1, \ldots, n-1$, $a_{1,n} \in M^Q \cdot A \subseteq A$ and $A = \langle x_n \rangle$, we obtain $\phi_1(g) \in A$.

Therefore,

\[
(4.4) \quad \phi_1(\langle h \rangle) = \phi_1(\langle x_n \rangle) = \langle x_n \rangle = \langle h \rangle.
\]

Consequently, we can proceed inductively to construct sequences 

\{b_{p,0}\}_{p \geq 1}, \text{ and } \{b_{p,i}\}_{p \geq 1} \text{ for } i = 1, \ldots, n \text{ with } b_{p,0} \in M^{Q+p-1} \text{ and } b_{p,i} \in M^{Q+p} \text{ for } i = 1, \ldots, n. \] By induction and Lemma 2.2, the generalizations of (4.3) and (4.4) hold, i.e. $g - u_p \cdot \varphi_p(f) \in M^{N+1+p}$ and $\varphi_p(\langle h \rangle) = \langle h \rangle$.

Again from Lemma 2.2, we obtain an automorphisms $(u, \varphi) \in K_A$ such that $g = u \cdot \varphi(f)$.

Case (2): Suppose $h = x_n + h_1(x_1, \ldots, x_{n-1})$.

Because $\psi : K[[x]] \to K[[x]]$, $x_i \mapsto x_i, (1 \leq i \leq n-1), x_n \mapsto h$ is an isomorphism, there is an inverse map $\psi^{-1} : K[[x]] \to K[[x]]$, $x_i \mapsto x_i, x_n \mapsto x_n - h_1(x_1, \ldots, x_{n-1})$.

Now let $Q = N - k \geq 1$, by assumption

\[
g - f \in M^{N+1} = M^{Q} \cdot \langle f \rangle + M^{Q} \cdot A \cdot \langle f_{x_n} \rangle + M^{Q+1} \cdot \{\{f_{x_i} \cdot h_{x_n} - f_{x_n} \cdot h_{x_i}; 1 \leq i \leq n-1\}).
\]
There exist $a_{1,0} \in \mathcal{M}^Q$, $a_{1,i} \in \mathcal{M}^{Q+1}$, and $a_{1,n} \in \mathcal{M}^Q \cdot \mathcal{A} \subset \mathcal{M}^{Q+1}$, $1 \leq i \leq n-1$ such that

$$g - f = a_{1,0} \cdot f + \sum_{1 \leq i \leq n-1} a_{1,i} \cdot (h_{x_i} \cdot f_{x_n} - h_{x_n} \cdot f_{x_i}) + a_{1,n} \cdot f_{x_n}$$

$$= a_{1,0} \cdot f + \sum_{i=1}^{n-1} a_{1,i} \cdot ((h_1)_{x_i} \cdot f_{x_n} - f_{x_i}) + a_{1,n} \cdot f_{x_n},$$

where $h_{x_n} = 1$ and $h_{x_i} = (h_1)_{x_i}$. One easily deduces that

$$\psi^{-1}(g) - \psi^{-1}(f) = \psi^{-1}(a_{1,0}) \cdot \psi^{-1}(f) +$$

$$+ \sum_{i=1}^{n-1} \psi^{-1}(a_{1,i}) \cdot \left[ \psi^{-1}((h_1)_{x_i}) \cdot \psi^{-1}(f_{x_n}) - \psi^{-1}(f_{x_i}) \right]$$

$$+ \psi^{-1}(a_{1,n}) \cdot \psi^{-1}(f_{x_n}) = \psi^{-1}(a_{1,0}) \cdot \psi^{-1}(f) -$$

$$- \sum_{i=1}^{n-1} \psi^{-1}(a_{1,i}) \cdot \left[ (-(h_1)_{x_i}) \cdot f_{x_n}(x_1, \ldots, x_{n-1}, x_n - h_1) \right] +$$

$$+ f_{x_i}(x_1, \ldots, x_{n-1}, x_n - h_1) +$$

$$+ \psi^{-1}(a_{1,n}) \cdot f_{x_n}(x_1, \ldots, x_{n-1}, x_n - h_1) = \psi^{-1}(a_{1,0}) \cdot \psi^{-1}(f) -$$

$$- \sum_{i=1}^{n-1} \psi^{-1}(a_{1,i}) \cdot \left( \psi^{-1}(f_{x_i}) - h_{x_i} \cdot \psi^{-1}(f_{x_n}) \right) + \psi^{-1}(a_{1,n}) \cdot \psi^{-1}(f_{x_n}),$$

i.e.,

$$(4.5) \quad \psi^{-1}(g) - \psi^{-1}(f) = \psi^{-1}(a_{1,0}) \cdot \psi^{-1}(f) -$$

$$- \sum_{i=1}^{n-1} \psi^{-1}(a_{1,i}) \cdot \left( \psi^{-1}(f_{x_i}) - h_{x_i} \cdot \psi^{-1}(f_{x_n}) \right) + \psi^{-1}(a_{1,n}) \cdot \psi^{-1}(f_{x_n}).$$

Let $b_{1,0} \doteq \psi^{-1}(a_{1,0}), \ b_{1,i} \doteq -\psi^{-1}(a_{1,i}), \ b_{1,n} \doteq \psi^{-1}(a_{1,n})$, then

$$\psi^{-1}(g) - \psi^{-1}(f) = b_{1,0} \cdot \psi^{-1}(f) + \sum_{i=1}^{n-1} b_{1,i} \cdot \left( \psi^{-1}(f) \right)_{x_i}$$

$$+ b_{1,n} \cdot \left( \psi^{-1}(f) \right)_{x_n},$$

where $b_{1,0} = \psi^{-1}(a_{1,0}) \in \mathcal{M}^Q$, $b_{1,i} = -\psi^{-1}(a_{1,i}) \in \mathcal{M}^{Q+1}$, $(i = 1, \ldots, n-1)$, and $b_{1,n} = \psi^{-1}(a_{1,n}) \in \mathcal{M}^Q \cdot \langle x_n \rangle$.

Therefore, we have

$$\psi^{-1}(g) - \psi^{-1}(f) \in \psi^{-1}(\mathcal{M}^{N+1}) = \mathcal{M}^{N+1}$$
4.8 so we get

\[ \psi^{-1}(g) - \psi^{-1}(f) \in M^Q \cdot \langle \psi^{-1}(f) \rangle + M^Q \cdot \langle x_n \rangle \cdot \langle \psi^{-1}(f) x_n \rangle \]

\[ + M^{Q+1} \cdot \langle \psi^{-1}(f) x_1, \ldots, \psi^{-1}(f) x_{n-1} \rangle. \]

Let \( \bar{v}_1 = 1 + b_{1,0} = 1 + \psi^{-1}(a_{1,0}) \in K[[x]]^* \) and \( \bar{\phi}_1 : K[[x]] \to K[[x]] : x_i \mapsto x_i + b_{1,i} = x_i - \psi^{-1}(a_{1,i}), (i = 1, \ldots, n - 1), x_n \mapsto x_n + b_{1,n} = x_n + \psi^{-1}(a_{1,n}), \) where \( a_{1,i} \in M^{Q+1} \) and \( a_{1,n} \in M^Q \cdot A. \)

We want to show that

\[ \psi^{-1}(g) - \bar{v}_1 \cdot \bar{\phi}_1 \left( \psi^{-1}(f) \right) \in M^{N+2} \]

and

\[ \psi^{-1}(g) - \bar{v}_1 \cdot \bar{\phi}_1 \left( \psi^{-1}(f) \right) \in M^{Q+1} \cdot \langle \psi^{-1}(f) \rangle + \]

\[ M^{Q+1} \cdot \langle x_n \rangle \cdot \left( \left( \psi^{-1}(f) \right) x_n \right) \]

\[ + M^{Q+2} \cdot \left( \left( \psi^{-1}(f) \right) x_1, \ldots, \left( \psi^{-1}(f) \right) x_{n-1} \right). \]

In fact, for \( \psi^{-1}(f) = \sum_{|\beta| \geq 0} l_\beta \cdot x^\beta, \)

\[ (4.6) \quad \psi^{-1}(f) \left( (x_1 + z_1), \ldots, (x_n + z_n) \right) \]

\[ = \sum_{|\beta| \geq 0} l_\beta \cdot \sum_{\gamma_1 = 0}^{\beta_1} \cdots \sum_{\gamma_n = 0}^{\beta_n} d_{\beta,\gamma} x^{\beta - \gamma} \cdot z^\gamma = \sum_{\alpha \in N_n} u_\alpha \cdot z^\alpha, \]

where \( u_\alpha = \sum_{|\beta| \geq 0, \beta \geq \alpha} l_\beta \cdot d_{\beta,\alpha} \cdot x^{\beta - \alpha}, \) it follows that \( \text{ord}(u_\alpha) \geq o - |\alpha|. \)

Applying \( \bar{\phi}_1 \) to \( \psi^{-1}(f) \) amounts to substituting \( z_j \) by \( -\psi^{-1}(a_{1,j}), j = 1, \ldots, n - 1, \) and \( z_n \) by \( -\psi^{-1}(a_{1,n}) \) in \( (4.8) \) so we get

\[ \bar{\phi}_1 \left( \psi^{-1}(f) \right) = \psi^{-1}(f) + \sum_{i=1}^{n-1} \left[ \psi^{-1}(f x_i) - \psi^{-1}(f x_n) \cdot (h_1)_{x_i} \right] \cdot \left( -\psi^{-1}(a_{1,i}) \right) + \psi^{-1}(a_{1,n}) \cdot \psi^{-1}(f x_n) + R, \]

where

\[ R = \sum_{|\alpha| \geq 2} d_\alpha \cdot (-\psi^{-1}(a_{1,1}))^{a_1} \cdots (-\psi^{-1}(a_{1,n-1}))^{a_{n-1}} \cdot (\psi^{-1}(a_{1,n}))^{a_n}. \]
Multiplying $\tilde{\phi}_1(f)$ by $\tilde{\nu}_1 = 1 + \psi^{-1}(a_{1,0})$ and using (4.5) we get
\[
\psi^{-1}(g) - \tilde{\nu}_1 \cdot \tilde{\phi}_1 \left( \psi^{-1}(f) \right)
\]
\[
= \psi^{-1}(g) - (1 + \psi^{-1}(a_{1,0})) \cdot
\]
\[
\left[ \psi^{-1}(f) + \sum_{i=1}^{n-1} \left( \psi^{-1}(f_{x_i}) - \psi^{-1}(f_{x_n}) \cdot h_{x_i} \right) \cdot \left( -\psi^{-1}(a_{1,i}) \right) + \psi^{-1}(a_{1,n}) \cdot \psi^{-1}(f_{x_n}) + R \right]
\]
\[
= \sum_{i=1}^{n-1} \left[ \psi^{-1}(f_{x_i}) - \psi^{-1}(f_{x_n}) \cdot (h_{1})_{x_i} \right] \cdot \left( -\psi^{-1}(a_{1,i}) \right) \cdot \psi^{-1}(a_{1,0})
\]
\[
+ \psi^{-1}(a_{1,n}) \cdot \psi^{-1}(f_{x_n}) \cdot \psi^{-1}(a_{1,0}) + (1 + \psi^{-1}(a_{1,0})) \cdot R
\]

Because $\text{ord}(h_1) \geq 1$ and $\text{ord} \left( \psi^{-1}(f_{x_i}) \right) \geq o - 1, (i = 1, \ldots, n - 1),$
\[
\text{ord} \left( \psi^{-1}(f_{x_i}) \cdot (-\psi^{-1}(a_{1,i})) \cdot \psi^{-1}(a_{1,0}) \right)
\]
\[
\geq o - 1 + (Q + 1) + Q = N + 2, \quad (i = 1, \ldots, n - 1),
\]
\[
\text{ord} \left( \psi^{-1}(f_{x_n}) \cdot (h_{1})_{x_i} \cdot (-\psi^{-1}(a_{1,i})) \cdot \psi^{-1}(a_{1,0}) \right)
\]
\[
\geq o - 1 + (Q + 1) + Q = N + 2, \quad (i = 1, \ldots, n - 1),
\]
\[
\text{ord} \left( \psi^{-1}(a_{1,n}) \cdot \psi^{-1}(f_{x_n}) \cdot \psi^{-1}(a_{1,0}) \right) \geq N + 2
\]

and
\[
\text{ord}(R) = \text{ord}(d_\alpha) + \sum_{i=1}^{n} \text{ord} \left( \psi^{-1}(a_{1,i}) \right) \cdot \alpha_i
\]
\[
\geq o - |\alpha| + (Q + 1) \cdot |\alpha| \geq N + 2,
\]

so $R \in \mathcal{M}^{N+2}$ and
\[
\psi^{-1}(g) - \tilde{\nu}_1 \cdot \tilde{\phi}_1 \left( \psi^{-1}(f) \right) \in \mathcal{M}^{N+2}.
\]

Hence we have proved (4.6).

Moreover, we have
\[
\tilde{\phi}_1(x_n) = (\tilde{\phi}_1)_n = x_n + \psi^{-1}(a_{1,n}) \in \langle x_n \rangle.
\]

Again by applying $\psi$ to (4.6), we get
\[
\psi \left( \psi^{-1}(g) \right) - \psi(\tilde{\nu}_1) \cdot \psi \left( \tilde{\phi}_1 \left( \psi^{-1}(f) \right) \right) \in \psi(\mathcal{M}^{N+2}) = \mathcal{M}^{N+2},
\]
i.e.
$$g - \psi(v_1) \cdot \psi \circ \tilde{\phi}_1 \circ \psi^{-1}(f) \in \mathcal{M}^{N+2}.$$ Moreover
$$\psi \circ \tilde{\phi}_1 \circ \psi^{-1}(h) = \psi\left(\tilde{\phi}_1(x_n)\right) = \psi\left(\tilde{\phi}_1(x_n)\right) = \langle h \rangle.$$ Consequently, let \( \phi_1 = \psi \circ \tilde{\phi}_1 \circ \psi^{-1} \) and \( v_1 = \psi(v_1) \), then
$$g - v_1 \cdot \phi_1(f) \in \mathcal{M}^{N+2}.$$ Since by assumption
$$\mathcal{M}^{N+2} = \mathcal{M}^{Q \cdot \langle f \rangle} + \mathcal{M}^{Q \cdot \mathcal{A} \cdot \langle f_{x_n} \rangle} + \mathcal{M}^{Q+1 \cdot \{ \{ f_{x_i} \cdot h_{x_n} - f_{x_n} \cdot h_{x_i} \} \mid 1 \leq i \leq n - 1 \}}$$
there exist \( d_{1,0} \in \mathcal{M}^{Q} \), \( d_{1,i} \in \mathcal{M}^{Q+1} \), and \( d_{1,n} \in \mathcal{M}^{Q \cdot \mathcal{A} \subset \mathcal{M}^{Q+1}, (1 \leq i \leq n - 1)} \) such that
$$g - v_1 \cdot \phi_1(f) = d_{1,0} \cdot f + \sum_{1 \leq j < n} d_{1,j} \cdot (h_{x_j} \cdot f_{x_n} - f_{x_n} \cdot h_{x_j}) + d_{1,n} \cdot f_{x_n}$$
$$= d_{1,0} \cdot f + \sum_{j=1}^{n-1} d_{1,j} \cdot ((h_1)_{x_j} \cdot f_{x_n} - f_{x_n}) + d_{1,n} \cdot f_{x_n}.$$ The proof of the following formula is similar to that of (4.5):
$$\psi^{-1}(g) - \psi^{-1}(v_1 \cdot \phi_1(f)) \in \mathcal{M}^{Q+1 \cdot \langle \psi^{-1}(f) \rangle} + \mathcal{M}^{Q+1 \langle x_n \rangle} \langle \langle \psi^{-1}(f) \rangle \rangle_{x_n}$$
$$+ \mathcal{M}^{Q+2} \langle \langle \psi^{-1}(f) \rangle \rangle_{x_1, \ldots, \langle \psi^{-1}(f) \rangle \rangle_{x_{n-1}}}.$$ Because
$$\psi^{-1}(g) - \psi^{-1}(v_1 \cdot \phi_1(f)) = \psi^{-1}(g) - \tilde{v}_1 \cdot \tilde{\phi}_1 \left(\psi^{-1}(f)\right),$$ we have proved (4.7).

Now we can proceed inductively to construct sequences \( b_{p,0} \triangleq \{ \psi^{-1}(a_{p,0}) \}_{p \geq 1} \), and \( b_{p,i} \triangleq \{ \psi^{-1}(a_{p,i}) \}_{p \geq 1} \) for \( i = 1, \ldots, n \), with \( b_{p,0} \in \mathcal{M}^{Q+p-1} \), \( b_{p,i} \in \mathcal{M}^{Q+p} \) for \( i = 1, \ldots, n - 1 \), and \( b_{p,n} \in \mathcal{M}^{Q+p-1} \cdot \langle x_n \rangle \).

By induction and Lemma 2.2, we can generalize (4.6) as:
$$\psi^{-1}(g) - \tilde{v}_p \cdot \tilde{\phi}_p \left(\psi^{-1}(f)\right) \in \mathcal{M}^{N+1+p}.$$
In the same way we also generalize (4.7) as:
\[
\psi^{-1}(g) - \tilde{u}_p \cdot \tilde{\varphi}_p(\psi^{-1}(f)) \in \mathcal{M}^{Q+p+1}(\psi^{-1}(f)) + \\
\mathcal{M}^{Q+p}(x_n) \cdot \left\langle (\psi^{-1}(f))_{x_n} \right\rangle + \mathcal{M}^{Q+p+1}(\langle (\psi^{-1}(f))_{x_1}, \ldots, (\psi^{-1}(f))_{x_{n-1}} \rangle).
\]
Meanwhile we have \( \tilde{\varphi}_p(\langle x_n \rangle) = \langle x_n \rangle \). Again by Lemma 2.2, we obtain \((\bar{u}, \tilde{\varphi}) \in \mathcal{K}\) such that
\[
\psi^{-1}(g) = \tilde{u} \cdot \tilde{\varphi}(\psi^{-1}(f)), \quad \text{and} \quad \tilde{\varphi}(\langle x_n \rangle) = \langle x_n \rangle.
\]
Therefore,
\[
g = \psi(\bar{u}) \cdot \psi(\tilde{\varphi}(\psi^{-1}(f))) = \psi(\bar{u}) \cdot (\psi \circ \tilde{\varphi} \circ \psi^{-1}(f)),
\]
and
\[
\psi \circ \tilde{\varphi} \circ \psi^{-1}(h) = \psi(\tilde{\varphi}(\psi^{-1}(h))) = \psi(\tilde{\varphi}(x_n)) = \psi(\langle x_n \rangle) = \langle h \rangle.
\]
Let \( u = \psi(\bar{u}) \) and \( \varphi = \psi \circ \tilde{\varphi} \circ \psi^{-1} \). Then we get \( g = u \cdot \varphi(f) \) with \((u, \varphi) \in \mathcal{K}_A\).

Case (3): Let \( h = H_1(x_1, \ldots, x_n) \cdot x_n + h_1(x_1, \ldots, x_{n-1}) \).

Combining the case (1) and the case (2), we get \((u, \varphi) \in \mathcal{K}_A\) such that \( g = u \cdot \varphi(f) \).

The proof for right equivalence goes along the same lines.

Let \( o = \text{ord}(f) \), the condition
\[
\mathcal{M}^{k+2} \subseteq \mathcal{M} \cdot j_A(f) \subseteq \mathcal{M}^{o+1}
\]
implies that \( k \geq o - 1 \) and that for any \( g \) with
\[
g - f \in \mathcal{M}^{N+1} = \mathcal{M}^{Q-1} \cdot \mathcal{M}^{k+2} \subseteq \mathcal{M}^{Q} \cdot j_A(f),
\]
where \( N = 2k - o + 2 \geq k + 1 \) and \( Q = N - k \geq 1 \), there are \( a_{1,i} \in \mathcal{M}^{Q+1} \) with
\[
g - f = \sum_{i=1}^{n-1} a_{1,i} (h_{x_i} \cdot f_{x_n} - h_{x_n} \cdot f_{x_i}) + a_{1,n} f_{x_n}
\]
\[
= \sum_{i=1}^{n-1} (-a_{1,i} h_{x_n} \cdot f_{x_i} + \sum_{i=1}^{n-1} a_{1,i} h_{x_i} \cdot f_{x_n} + a_{1,n} \cdot f_{x_n}).
\]
We can then define \( \phi_1 \) as above and see that
\[
g - \phi_1(f) = r \in \mathcal{M}^{N+2}.
\]
Going on by induction and applying Lemma 2.2, we get an automorphism \( \varphi \in \mathcal{R}_A \) such that \( g = \varphi(f) \).
Corollary 4.6. Let $0 \neq f \in \mathcal{M}^2 \subseteq K[[x]]$.

1. If $\mu_A(f) < \infty$, then $f$ is $(2\mu_A(f) - \text{ord}(f)) - \mathcal{R}_A$-determined.
2. If $\tau_A(f) < \infty$, then $f$ is $(2\tau_A(f) - \text{ord}(f)) - \mathcal{K}_A$-determined.

Combining Theorem 4.3 and Corollary 4.6, we obtain:

Theorem 4.7. Let $0 \neq f \in \mathcal{M} \subseteq K[[x]]$ be a power series.

1. $f$ is a relative $\mathcal{A}$-isolated singularity if and only if $f$ is finitely $\mathcal{R}_A$-determined.
2. $\mathcal{R}_f$ is a relative $\mathcal{A}$-isolated hypersurface singularity if and only if $f$ is finitely $\mathcal{K}_A$-determined.

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