Title:
Some results on value distribution of the difference operator

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SOME RESULTS ON VALUE DISTRIBUTION OF THE DIFFERENCE OPERATOR

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Abstract. In this article, we consider the uniqueness of the difference monomials $f^m(z)f(z+c)$. Suppose that $f(z)$ and $g(z)$ are transcendental meromorphic functions with finite order and $E_k(1, f^m(z)f(z+c)) = E_k(1, g^m(z)g(z+c))$. Then we prove that if one of the following holds (i) $n \geq 14$ and $k \geq 3$, (ii) $n \geq 16$ and $k = 2$, (iii) $n \geq 22$ and $k = 1$, then $f(z) \equiv t_1 g(z)$ or $f(z)g(z) = t_2$, for some constants $t_1$ and $t_2$ that satisfy $t_1^{n+1} = 1$ and $t_2^{n+1} = 1$. We generalize some previous results of Qi et. al.

Keywords: Meromorphic functions, difference equations, uniqueness, finite order.

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1. Introduction and main results

In this article, we assume that the reader is familiar with the fundamental results and the standard notations of the Nevanlinna theory (see, e.g., [8,18]). Let $f(z)$ and $g(z)$ be two non-constant meromorphic functions in the complex plane. By $S(r,f)$, we denote any quantity satisfying $S(r,f) = o(T(r,f))$ as $r \to \infty$, possibly outside a set of finite logarithmic measure. Then the meromorphic function $a$ is called a small function of $f(z)$, if $T(r,a) = S(r,f)$. If $f(z) - a$ and $g(z) - a$ have same zeros, counting multiplicity (ignoring multiplicity), then we say $f(z)$ and $g(z)$ share the small function $a$ CM (IM).

Let $a$ be a finite complex number, and $k$ be a positive integer. We denote by $N_k(r, \frac{1}{T-a})$ the counting function for the zeros of $f(z)-a$ with multiplicity $\leq k$, and by $\bar{N}_k(r, \frac{1}{T-a})$ the corresponding one for which multiplicity is not counted. Let $N_k(r, \frac{1}{T-a})$ be the counting function for the zeros of $f(z) - a$ with multiplicity $\geq k$, and $\bar{N}_k(r, \frac{1}{T-a})$ be the corresponding one for which multiplicity is
not counted. Moreover, we set \( N_k(r, \frac{1}{f^n}) = N(r, \frac{1}{f^n}) + N_2(r, \frac{1}{f^n}) + \cdots + N_k(r, \frac{1}{f^n}) \). In the same way, we can define \( N_k(r, f) \).

Currently, many articles have focused on value distribution in difference analogues of meromorphic functions (see, e.g., [1, 2, 5–7, 9–17, 19]). In particular, there has been an increasing interest in studying the uniqueness problems related to meromorphic functions and their shifts or their difference operators (see, e.g., [1, 9, 11–15]). Our aim in this article is to investigate the uniqueness problems of difference monomials of meromorphic functions.

In 2010, Qi et al. [16] studied the uniqueness of the difference monomials and obtained the following result:

**Theorem 1.1.** Let \( f(z) \) and \( g(z) \) be transcendental entire functions with finite order, \( c \) a non-zero complex constant; and \( n \geq 6 \) an integer. If \( E(1, f^n(z)f(z + c)) = E(1, g^n(z)g(z + c)) \), then \( f(z) \equiv t_1g(z) \) or \( f(z)g(z) = t_2 \), for some constants \( t_1 \) and \( t_2 \) that satisfy \( t_1^{n+1} = 1 \) and \( t_2^{n+1} = 1 \).

In this paper, we consider the case of meromorphic functions of Theorem A. Our result can be stated as follows:

**Lemma 1.2.** Let \( c \in \mathbb{C} \setminus \{0\} \). Let \( f(z) \) and \( g(z) \) be two transcendental meromorphic functions with finite order, and \( n \geq 14 \), \( k \geq 3 \) be two positive integers. If \( E_k(1, f^n(z)f(z + c)) = E_k(1, g^n(z)g(z + c)) \), then \( f(z) \equiv t_1g(z) \) or \( f(z)g(z) = t_2 \), for some constants \( t_1 \) and \( t_2 \) that satisfy \( t_1^{n+1} = 1 \) and \( t_2^{n+1} = 1 \).

In the previous theorem we considered the case \( k \geq 3 \). The following two theorems are about the case \( k \leq 2 \).

**Theorem 1.3.** Let \( c \in \mathbb{C} \) and \( n \geq 16 \) be an integer. Let \( f(z) \) and \( g(z) \) be two transcendental meromorphic functions with finite order. If \( E_2(1, f^n(z)f(z + c)) = E_2(1, g^n(z)g(z + c)) \), then \( f(z) \equiv t_1g(z) \) or \( f(z)g(z) = t_2 \), for some constants \( t_1 \) and \( t_2 \) that satisfy \( t_1^{n+1} = 1 \) and \( t_2^{n+1} = 1 \).

**Theorem 1.4.** Let \( c \in \mathbb{C} \) and \( n \geq 22 \) be an integer. Let \( f(z) \) and \( g(z) \) be two transcendental meromorphic functions with finite order. If \( E_4(1, f^n(z)f(z + c)) = E_4(1, g^n(z)g(z + c)) \), then \( f(z) \equiv t_1g(z) \) or \( f(z)g(z) = t_2 \), for some constants \( t_1 \) and \( t_2 \) that satisfy \( t_1^{n+1} = 1 \) and \( t_2^{n+1} = 1 \).

### 2. Preliminary lemmas

Before proceeding to the actual proofs, we recall a few lemmas that play an important role in the reasoning.

**Lemma 2.1.** [3] Let \( f \) and \( g \) be two meromorphic functions, and let \( k \) be a positive integer. If \( E_k(1, f) = E_k(1, g) \), then one of the following cases must occur:
1. \[ T(r,f) + T(r,g) \leq N_2(r,f) + N_2(r,\frac{1}{f}) + N_2(r,g) + N_2(r,\frac{1}{g}) \]
\[+N\left(r,\frac{1}{f-1}\right) + N\left(r,\frac{1}{g-1}\right) - N_1\left(r,\frac{1}{f-1}\right) + N_{k+1}\left(r,\frac{1}{f-1}\right) \]
\[= N_{k+1}\left(r,\frac{1}{g-1}\right) + S(r,f) + S(r,g); \]
\[ (2.1) \]

2. \[ f = \frac{(b + 1)g + (a - b - 1)}{bg + (a - b)}, \]
where \(a \neq 0\), \(b\) are two constants.

Lemma 2.2. Let \(f(z)\) be a nonconstant finite order meromorphic function and let \(c \neq 0\) be an arbitrary complex number. Then
\[ T(r, f(z + |c|)) = T(r, f(z)) + S(r,f). \]

Remark 2.3. It is shown in [4, p. 66], that for \(c \in \mathbb{C} \setminus \{0\}, \)
\[ (1 + o(1))T(r - |c|, f(z)) \leq T(r, f(z + c)) \leq (1 + o(1))T(r + |c|, f(z)) \]
hold as \(r \to \infty\), for a general meromorphic function. By this and Lemma 2.2, we obtain
\[ T(r, f(z + c)) = T(r, f(z)) + S(r,f). \]

Lemma 2.4. Let \(f(z)\) and \(g(z)\) be two meromorphic functions with finite order, \(n \geq 8\) a positive integer, and let \(F = f^n(z)f(z + c)\) and \(G = g^n(z)g(z + c)\). If
\[ F = \frac{(b + 1)G + (a - b - 1)}{bg + (a - b)}, \]
where \(a \neq 0\), \(b\) are two constants, then \(f(z) = t_1g(z)\) or \(f(z)g(z) = t_2\), for some constants \(t_1\) and \(t_2\) that satisfy \(t_1^{n+1} = 1\) and \(t_2^{n+1} = 1\).

Proof of Lemma 2.3. Remark 2.3 yields that
\[ T(r, F) = T(r, f^n(z)f(z + c)) + S(r,f) \]
\[ \leq T(r, f^n(z)) + T(r, f(z + c)) + S(r,f) \]
\[ = (n + 1)T(r, f) + S(r,f). \]

On the other hand, together the first main Theorem with Remark 2.3, we obtain
\[ nT(r,f) = T(r, f^n(z)) + S(r,f) \]
\[ \leq T(r, f^n(z)f(z + c)) + T\left(r, \frac{1}{f(z + c)}\right) + S(r,f) \]
\[ = T(r, f(z)) + T(r, F(z)) + S(r,f) \]
\[ (2.7) \]
that is,
(2.8) \[ T(r, F) \geq (n - 1)T(r, f) + S(r, f) \]

Hence, (2.4) and (2.6) yield that
(2.9) \[ S(r, F) = S(r, f). \]

Similarly, we obtain
(2.10) \[ T(r, G) \geq (n - 1)T(r, g) + S(r, g), \]

and
(2.11) \[ S(r, G) = S(r, g). \]

Set \( I_1 = \{ r : T(r, g) \geq T(r, f) \} \subseteq (0, \infty), \) and \( I_2 = (0, \infty) \setminus I_1. \) Then there is at least one \( I_i (i = 1, 2) \) such that \( I_i \) has infinite logarithmic measure. Without loss of generality, we may suppose that \( I_1 \) has infinite logarithmic measure. We break the rest of the proof into three cases. □

Case 1. \( b \neq 0, -1. \) If \( a - b - 1 \neq 0, \) then we know from 2.3

(2.12) \[ N\left(r, \frac{1}{F}\right) = N\left(r, \frac{1}{G - \frac{a-b-1}{b+1}}\right). \]

Together with the first main theorem, the second main theorem with Remark 2.3, 2.8 and 2.12, we obtain

\[
(n - 1)T(r, g) \leq T(r, G) + S(r, g) \\
\quad \leq N\left(r, \frac{1}{G}\right) + N(r, G) + N\left(r, \frac{1}{G - \frac{a-b-1}{b+1}}\right) + S(r, G) + S(r, g) \\
\quad = N\left(r, \frac{1}{G}\right) + N(r, G) + N\left(r, \frac{1}{F}\right) + S(r, g) \\
\quad \leq N\left(r, \frac{1}{g}\right) + N\left(r, \frac{1}{g(z+c)}\right) + N(r, g) + N(r, g(z+c)) \\
\quad \quad + N\left(r, \frac{1}{f(z+c)}\right) + S(r, g) \\
\quad \leq 4T(r, g) + 2T(r, f) + S(r, g) \\
\quad \leq 6T(r, g) + S(r, g), \quad r \in I_1
\]

which is impossible, since \( n \geq 8. \) Hence, we obtain \( a - b - 1 = 0, \) so

\[ F(z) = \frac{(b+1)G(z)}{bG(z) + 1}. \]
Using the similar method as above, we obtain
\[
(n - 1)T(r, g) \leq T(r, G) + S(r, g)
\]
\[
\leq \mathcal{N}\left(r, \frac{1}{G}\right) + \mathcal{N}(r, G) + \mathcal{N}\left(r, \frac{1}{G + \varepsilon}\right) + S(r, G)
\]
\[
= \mathcal{N}\left(r, \frac{1}{G}\right) + \mathcal{N}(r, G) + \mathcal{N}(r, F) + S(r, G)
\]
\[
\leq 6T(r, g) + S(r, g), \quad r \in I_1
\]
which is a contradiction, since \( n \geq 8 \).

Case 2. \( b = -1, a \neq -1 \). By 2.3, we have
\[
(2.15) \quad F = \frac{a}{a + 1 - G}.
\]

Similarly, we get a contradiction, hence, we obtain \( a = -1 \). So, we get \( FG = 1 \), that is \( f^n(z)f(z + c)g^n(z)g(z + c) = 1 \). Set \( H(z) = f(z)g(z) \). Suppose that \( H(z) \) is not a constant. Then we obtain
\[
(2.16) \quad H^n(z)H(z + c) = 1.
\]

Remark 2.3, the first main Theorem and 2.16 imply that
\[
(2.17) \quad nT(r, H(z)) = T(r, H^n(z)) = T\left(r, \frac{1}{H(z + c)}\right) = T(r, H(z)) + S(r, H).
\]

Hence \( H(z) \) must be a nonzero constant, since \( n \geq 8 \). Set \( H(z) = t_1 \), by 2.16, we know \( t_1^{n+1} = 1 \). Thus, \( f(z)g(z) = t_1 \), where \( t_1^{n+1} = 1 \).

Case 3. \( b = 0, a \neq 1 \). By 2.3, we obtain
\[
F = \frac{G + a - 1}{a}.
\]

Similarly, we get a contradiction, hence we obtain \( a = 1 \), so we get \( F = G \), that is
\[
f^n(z)f(z + c) = g^n(z)g(z + c).
\]
Let \( H(z) = \frac{f(z)}{g(z)} \), using the similar method as above, we also obtain that \( H(z) \) must be a nonzero constant. Thus, we have \( f = t_2g \), where \( t_2^{n+1} = 1 \).

3. **Proof of Theorem 1.1**

Let \( F(z) = f^n(z)f(z + c) \) and \( G(z) = g^n(z)g(z + c) \). Since \( k \geq 3 \), we have
\[
\mathcal{N}\left(r, \frac{1}{F - 1}\right) + \mathcal{N}\left(r, \frac{1}{G - 1}\right) - N_{1,1}\left(r, \frac{1}{F - 1}\right) + \mathcal{N}_{(k+1)}\left(r, \frac{1}{F - 1}\right)
\]
\[
+ \mathcal{N}_{(k+1)}\left(r, \frac{1}{G - 1}\right) \leq \frac{1}{2}N\left(r, \frac{1}{F - 1}\right) + \frac{1}{2}N\left(r, \frac{1}{G - 1}\right)
\]
\[
(3.1) \quad \leq \frac{1}{2}T(r, F) + \frac{1}{2}T(r, G) + S(r, f) + S(r, g).
\]
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2.1 and 3.1 give that

\[ T(r, F) + T(r, G) \leq 2 \left\{ N_2 \left( r, \frac{1}{F} \right) + N_2(r, F) + N_2 \left( r, \frac{1}{G} \right) + N_2(r, G) \right\} + S(r, f) + S(r, g). \]

Together the definition of \( F \), the first main Theorem with Remark 2.3, we have

\[ N_2 \left( r, \frac{1}{F} \right) \leq 2N \left( r, \frac{1}{f} \right) + N \left( r, \frac{1}{f(z+c)} \right) + S(r, f) \]

(3.3)

\[ \leq 3T(r, f) + S(r, f). \]

Similarly,

\[ N_2 \left( r, \frac{1}{G} \right) \leq 3T(r, g) + S(r, f), \]

(3.4)

\[ N_2(r, F) \leq 3T(r, f) + S(r, f), \]

(3.5)

\[ N_2(r, G) \leq 3T(r, g) + S(r, f). \]

(3.2)-(3.6) yield that

\[ T(r, F) + T(r, G) \leq 12(T(r, f) + T(r, g)) + S(r, f) + S(r, g). \]

Then, by (2.6), (2.8) and (3.7), we obtain

\[ (n - 13)|T(r, f) + T(r, g)| \leq S(r, f) + S(r, g), \]

which is a contradiction since \( n \geq 14 \). Hence, by Lemma 2.1, we have \( F = (b + 1)G + \frac{(a-b-1)}{(4 + a-b)} \), where \( a \neq 0, b \) are two constants. By Lemma 2.3, we get \( f(z) \equiv t_1 g(z) \) or \( f(z)g(z) = t_2 \), for some constants \( t_1 \) and \( t_2 \) that satisfy \( t_1^{n+1} = 1 \) and \( t_2^{n+1} = 1 \).

4. Proof of Theorem 1.2

Note that

\[ N \left( r, \frac{1}{F-1} \right) + N \left( r, \frac{1}{G-1} \right) - N_{11} \left( r, \frac{1}{F-1} \right) + \frac{1}{2} N_{10} \left( r, \frac{1}{F-1} \right) + \frac{1}{2} N_{10} \left( r, \frac{1}{G-1} \right) \]

\[ \leq \frac{1}{2} N \left( r, \frac{1}{F-1} \right) + \frac{1}{2} N \left( r, \frac{1}{G-1} \right) \]

(4.1)

\[ \leq \frac{1}{2} T(r, F) + \frac{1}{2} T(r, G) + S(r, f) + S(r, g). \]

Then we obtain from 2.1 and 4.1

\[ T(r, F) + T(r, G) \leq 2 \left\{ N_2 \left( r, \frac{1}{F} \right) + N_2(r, F) + N_2 \left( r, \frac{1}{G} \right) + N_2(r, G) \right\} \]

\[ + N_{10} \left( r, \frac{1}{F-1} \right) + N_{10} \left( r, \frac{1}{G-1} \right) + S(r, f) + S(r, g) \]
Obviously, combining the first main Theorem and Remark 2.3, we have
\[
\mathcal{N}(r, \frac{1}{F-1}) \leq \frac{1}{2} \mathcal{N}(r, \frac{F'}{F})
\]
\[
= \frac{1}{2} \mathcal{N}(r, \frac{F'}{F}) + S(r, f)
\]
\[
\leq \frac{1}{2} \mathcal{N}(r, F) + \frac{1}{2} \mathcal{N}(r, \frac{1}{F}) + S(r, f)
\]
\[
\leq \frac{1}{2} \left[ \mathcal{N}(r, f(z)) + \mathcal{N}(r, f(z + c)) + \mathcal{N}(r, \frac{1}{f(z)}) + \mathcal{N}(r, \frac{1}{f(z + c)}) \right] + S(r, f) \leq 2T(r, f) + S(r, f).
\]

Similarly, we obtain
\[
\mathcal{N}(r, \frac{1}{G-1}) \leq 2T(r, g) + S(r, f)
\]

Suppose that
\[
T(r, F) + T(r, G) \leq 2 \left\{ N_2(r, \frac{1}{F}) + N_2(r, F) + N_2(r, \frac{1}{G}) + N_2(r, G) \right\}
\]
\[
+ \mathcal{N}(3, \frac{1}{F-1}) + \mathcal{N}(3, \frac{1}{G-1}) + S(r, f) + S(r, g).
\]

Then we have from 2.8, 2.10, 3.3-3.6 and 4.2-4.4
\[
(n - 1)T(r, f) + (n - 1)T(r, g) \leq T(r, F) + T(r, G)
\]
\[
\leq 14T(r, f) + 14T(r, g) + S(r, f) + S(r, g),
\]

which is a contradiction, since \( n \geq 16 \). By Lemma 2.1, we obtain that
\[
F = (b + 1)G + \frac{(a-b-1)}{\alpha \gamma a - a - b},
\]
where \( a \neq 0, b \) are two constants. By Lemma 2.3, we get \( f(z) = t_1g(z) \) or \( f(z)g(z) = t_2 \), for some constants \( t_1 \) and \( t_2 \) that satisfy \( t_1^{n+1} = 1 \) and \( t_2^{n+1} = 1 \).

5. Proof of Theorem 1.3

Since
\[
\mathcal{N}(r, \frac{1}{F-1}) + \mathcal{N}(r, \frac{1}{G-1}) - N_{11}(r, \frac{1}{F-1})
\]
\[
\leq \frac{1}{2} \mathcal{N}(r, \frac{1}{F-1}) + \frac{1}{2} \mathcal{N}(r, \frac{1}{G-1})
\]
\[
\leq \frac{1}{2} T(r, F) + \frac{1}{2} T(r, G) + S(r, f) + S(r, g).
\]

Then (2.1) becomes
\[
T(r, F) + T(r, G) \leq 2 \left\{ N_2(r, \frac{1}{F}) + N_2(r, F) + N_2(r, \frac{1}{G}) + N_2(r, G) + \mathcal{N}(2, \frac{1}{F-1}) + \mathcal{N}(2, \frac{1}{G-1}) \right\} + S(r, f) + S(r, g).
\]
Combining the first main Theorem and Remark 2.3, we obtain
\[ N(2(r, \frac{1}{F-1})) \leq N(r, \frac{F}{F'}) \]
\[ = N(r, \frac{F''}{F}) + S(r, f) \]
\[ \leq N(r, F) + N(r, \frac{1}{F}) + S(r, f) \]
\[ \leq N(r, f(z)) + N(r, f(z + c)) + N(r, \frac{1}{f(z)}) + N(r, \frac{1}{f(z + c)}) \]
\[ + S(r, f) \]
(5.2)

Similarly, we get
\[ N(2(r, \frac{1}{G-1})) \leq 4T(r, f) + S(r, f). \]
(5.3)

Suppose that
\[ T(r, F) + T(r, G) \leq 2\left( N_2(r, \frac{1}{F}) + N_2(r, F) + N_2(r, \frac{1}{G}) + N_2(r, G) \right) \]
\[ + N(2(r, \frac{1}{F-1})) + N_2(r, \frac{1}{G-1}) \right) + S(r, f) + S(r, g). \]
(5.4)

Then we obtain from 2.8, 2.10, 3.3-3.6 and 5.2-5.4
\[ (n - 1)T(r, f) + (n - 1)T(r, g) \leq T(r, F) + T(r, G) \]
\[ \leq 20T(r, f) + 20T(r, g) + S(r, f) + S(r, g), \]

which is impossible, since \( n \geq 22 \). By Lemma 2.1, we obtain that \( F = (b + 1)G + \frac{(a-b-1)}{M\alpha+n+b} \), where \( a \neq 0 \), \( b \) are two constants. By Lemma 2.3, we get \( f(z) \equiv t_1g(z) \) or \( f(z)g(z) = t_2 \), for some constants \( t_1 \) and \( t_2 \) that satisfy \( t_1^{n+1} = 1 \) and \( t_2^{n+1} = 1 \).

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