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SOME PROPERTIES OF EXTENDED MULTIPLIER TRANSFORMATIONS TO THE CLASSES OF MEROMORPHIC MULTIVALENT FUNCTIONS

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Abstract. In this paper, we introduce new classes $\sum_{k,p,n}(\alpha, m, \lambda, l, \rho)$ and $\mathcal{T}_{k,p,n}(\alpha, m, \lambda, l, \rho)$ of p-valent meromorphic functions defined by using the extended multiplier transformation operator. We use a strong convolution technique and derive inclusion results. A radius problem and some other interesting properties of these classes are discussed.
Keywords: Multivalent functions, analytic functions, meromorphic functions, multiplier transformations, linear operator, functions with positive real part, Hadamard product (or Convolution).
MSC(2010): Primary: 30C45; Secondary: 30C50.

1. Introduction

Let $\sum_{p,n}$ denote the class of functions of the form

$$f(z) = \frac{1}{z^p} + \sum_{t=n}^{\infty} a_t \, z^t, \qquad (p \in \mathbb{N} = \{1, 2, ...\}; n > -p), \tag{1.1}$$

which are analytic in the punctured unit disk

$$E^* = \{z \ : z \in \mathbb{C} \text{ and } 0 < |z| < 1\} = E \backslash \{0\}.$$

For two functions $f_j(z) \in \sum_{p,n}$ $(j = 1, 2)$, given by

$$f_j(z) = \frac{1}{z^p} + \sum_{t=n}^{\infty} a_{t,j} \, z^t, \qquad (j = 1, 2), \tag{1.2}$$

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we define the Hadamard product (or convolution) of $f_1(z)$ and $f_2(z)$ by

$$(f_1 * f_2)(z) = \frac{1}{z^p} + \sum_{t=n}^{\infty} a_{t,1}\ a_{t,2} z^k = (f_2 * f_1)(z). \tag{1.3}$$

Let $P_k(\rho)$ be the class of functions $p(z)$ analytic in E with $p(0) = 1$ and

$$\int_0^{2\pi} \left| \frac{\Re p(z) - \rho}{1 - \rho} \right| d\theta \leq k\pi,\ z = re^{i\theta}, \tag{1.4}$$

where $k \geqslant 2$ and $0 \leq \rho < 1$. This class was introduced by Padmanabhan et al. in [13]. We note that $P_k(0) = P_k$, see Pinchuk [15], $P_2(\ \rho) = P(\rho)$, the class of analytic functions with positive real part greater than ρ and $P_2(0) = P$, the class of functions with positive real part. We can write (1.4) as

$$p(z) = \frac{1}{2} \int_0^{2\pi} \frac{1 + (1 - 2\rho)ze^{-it}}{1 - ze^{-it}} d\mu(t),$$

where $\mu(t)$ is a function with bounded variation on $[0, 2\pi]$ such that

$$\int_0^{2\pi} d\mu(t) = 2, \quad \text{and} \quad \int_0^{2\pi} |d\mu(t)| \leq k.$$

From (1.4) we can easily deduce that $p(z) \in P_k(\rho)$ if, and only if, there exists $p_1(z)$, $p_2(z) \in P(\rho)$ such that for $z \in E$,

$$p(z) = \left(\frac{k}{4} + \frac{1}{2}\right) p_1(z) - \left(\frac{k}{4} - \frac{1}{2}\right) p_2(z). \tag{1.5}$$

For $l > 0$, $\lambda \geq 0$ and $m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, Ashwah [5] defined the multiplier transformation $J_p^m(\lambda, l)$ of functions $f \in \sum_{p,n}$ by

$$J_p^m(\lambda, l)f(z) = \frac{1}{z^p} + \sum_{t=n}^{\infty} \left(\frac{l + \lambda(k + p)}{l}\right)^m a_t\ z^t \quad (l > 0;\ \lambda \geq 0; z \in E^*). \tag{1.6}$$

Obviously, we have

$$J_p^{m_1}(\lambda, l)(J_p^{m_2}(\lambda, l)f(z)) = J_p^{m_1+m_2}(\lambda, l)f(z) = J_p^{m_2}(\lambda, l)(J_p^{m_1}(\lambda, l)f(z)), \tag{1.7}$$

for all positive integers m_1 and m_2.

We note that

(*i*) $J_1^m(1, l)f(z) = I(m, l)f(z)$, see Cho et al [3, 4];

(*ii*) $J_1^m(1, 1)f(z) = I^m f(z)$, see Uralegaddi and Somanatha [21].

(*iii*) $J_1^m(\lambda, 1)f(z) = D_{\lambda,p}^m f(z)$, see Al-Oboudi and Al-Zkero [1].

Ashwa [6] defined the integral operator $\mathcal{L}_p^m(\lambda,l)f(z)$ as follows:

$$\mathcal{L}_p^0(\lambda,l)f(z) = f(z),$$
$$\mathcal{L}_p^1(\lambda,l)f(z) = \left(\frac{l}{\lambda}\right) z^{-p-\left(\frac{l}{\lambda}\right)} \int_0^z t^{\left(\frac{l}{\lambda}+p-1\right)} f(t)dt \quad (f \in \sum\nolimits_{p,n} \; ; z \in E^*),$$
$$\mathcal{L}_p^2(\lambda,l)f(z) = \left(\frac{l}{\lambda}\right) z^{-p-\left(\frac{l}{\lambda}\right)} \int_0^z t^{\left(\frac{l}{\lambda}+p-1\right)} \mathcal{L}_p^1(\lambda,l)f(t)dt \quad (f \in \sum\nolimits_{p,n} \; ; z \in E^*),$$

and, in general,

$$\mathcal{L}_p^m(\lambda,l)f(z) = \left(\frac{l}{\lambda}\right) z^{-p-\left(\frac{l}{\lambda}\right)} \int_0^z t^{\left(\frac{l}{\lambda}+p-1\right)} \mathcal{L}_p^{m-1}(\lambda,l)f(t)dt$$

$$= \mathcal{L}_p^1(\lambda,l)\left(\frac{1}{z^p(1-z)}\right) * \mathcal{L}_p^1(\lambda,l)\left(\frac{1}{z^p(1-z)}\right) * \ldots \tag{1.8}$$
$$*\mathcal{L}_p^1(\lambda,l)\left(\frac{1}{z^p(1-z)}\right) * f(z)$$
$$\lfloor - \; - \; - \; - \; - m - times - \; - \; - \; - \rceil$$
$$(f \in \sum\nolimits_{p,n} \; ; \; m \in \mathbb{N}_{\not{\!\!\!\!0}} \; ; \; z \in E^*). \tag{1.9}$$

We note that if $f(z) \in \sum_{p,n}$, then from (1.1) and (1.8), we have

$$\mathcal{L}_p^m(\lambda,l)f(z) = \frac{1}{z^p} + \sum_{t=n}^{\infty}\left(\frac{l}{l+\lambda(k+p)}\right)^m a_t \; z^t$$
$$(l > 0; \; \lambda \geq 0; \; p \in \mathbb{N}; \; m \in \mathbb{N}_0 \; ; \; z \in E^*). \tag{1.10}$$

From (1.9), Ashwa [6] obtained the following properties:

$$\lambda z(\mathcal{L}_p^{m+1}(\lambda,l)f(z))' = l\mathcal{L}_p^m(\lambda,l)f(z) - (l+\lambda p)\mathcal{L}_p^{m+1}(\lambda,l)f(z) \quad (\lambda > 0). \tag{1.10}$$

We note that:

$$\mathcal{L}_p^m(1,\beta)f(z) = P_{p,\beta}^{\alpha}f(z), \text{ (see Aqlan et al. [2])}$$
$$\mathcal{L}_1^{\alpha}(1,\beta)f(z) = \mathcal{L}_{p,l}^m f(z) \quad \text{(see Lashin [7])}.$$

Also, we note that (see Ashwah [6])

(i) $\mathcal{L}_p^m(1,l)f(z) = \mathcal{L}_{p,l}^m f(z)$, where $\mathcal{L}_{p,l}^m(\lambda,l)f(z)$ is given by (1.9).

(ii) $\mathcal{L}_p^m(1,1)f(z) = \mathcal{L}_p^m f(z)$, where $\mathcal{L}_p^m f(z)$ is given by (1.9).

Definition 1.1. Let $f(z) \in \sum_{p,n}$. Then, $f \in \sum_{k,p,n}(\alpha,m,\lambda,l,\rho)$ if, and only if,

$$\left\{(1-\alpha)(z^p\mathcal{L}_p^m(\lambda,l)f(z)) + \frac{\alpha}{p}z^{p+1}(\mathcal{L}_p^m(\lambda,l)f(z))'\right\} \in P_k(\rho),$$

where α is a complex number, $k \geq 2$, $z \in E$ and $0 \leq \rho < p$.

Definition 1.2. Let $f \in \sum_{p,n}$. Then, $f \in \mathcal{T}_{k,p,n}(\alpha, m, \lambda, l, \rho)$ if, and only if,

$$\left\{(1-\alpha)(z^p \mathcal{L}_p^{m+1}(\lambda, l) f(z)) + \alpha(z^p \mathcal{L}_p^m(\lambda, l) f(z))\right\} \in P_k(\rho),$$

where $\alpha > 0$, $k \geq 2$, $z \in E$, and $0 \leq \rho < p$.

In this paper, we introduce new classes of p-valent meromorphic functions defined by using the extended multiplier transformation operator. We use a strong convolution technique and derive inclusion results, a radius problem and some other interesting properties of these classes are discussed as well.

The interested reader are referred to the research works [5, 6, 8, 9, 10, 18, 19, 20].

2. **Preliminary results**

To establish our main results we need the following Lemmas.

Lemma 2.1. [16]

If $p(z)$ is analytic in E with $p(0) = 1$, and if λ_1 is a complex number satisfying $\Re(\lambda_1) \geq 0$ $(\lambda_1 \neq 0)$, then

$$\Re\Big\{p(z) + \lambda_1 z p^{'}(z)\Big\} > \beta \qquad (0 \leq \beta < 1).$$

Implies

$$\Re p(z) > \beta + (1-\beta)(2\gamma - 1),$$

where γ is given by

$$\gamma = \gamma(\Re\lambda_1) = \int_0^1 (1 + t^{\Re\ \lambda})^{-1} dt,$$

which is an increasing function of $\Re\lambda_1$ and $\frac{1}{2} \leq \gamma < 1$. The estimate is sharp in the sense that the bound cannot be improved.

Lemma 2.2. [17]

If $p(z)$ is analytic in E, $p(0) = 1$ and $\Re p(z) > \frac{1}{2}$, $z \in E$, then for any function F analytic in E, the function $p * F$ takes values in the convex hull of the image of E under F.

Lemma 2.3. [14]

Let $p(z) = 1 + b_1 z + b_2 z^2 + ... \in P(\rho)$. Then,

$$\Re p(z) \geq 2\rho - 1 + \frac{2(1-\rho)}{1+|z|}.$$

3. Main results

Theorem 3.1. *Let* $\Re\alpha > 0$. *Then,*

$$\sum_{k,p,n}(\alpha, m, \lambda, l, \rho) \subset \sum_{k,p,n}(0, m, \lambda, l, \rho_1),$$

where ρ_1 *is given by*

$$\rho_1 = \rho + (1-\rho)(2\gamma - 1), \tag{3.1}$$

and

$$\gamma = \int_0^1 \left(1 + t^{\Re\frac{\alpha}{p}}\right)^{-1} dt.$$

Proof. Let $f \in \sum_{k,p,n}(\alpha, m, \lambda, l, \rho)$, and set

$$z^p(\mathcal{L}_p^m(\lambda, l)f(z)) = p(z) = \left(\frac{k}{4} + \frac{1}{2}\right)p_1(z) - \left(\frac{k}{4} - \frac{1}{2}\right)p_2(z). \tag{3.2}$$

Then, $p(z)$ is analytic in E with $p(0) = 1$. After a simple computations, we have

$$\left\{(1-\alpha)(z^p\mathcal{L}_p^m(\lambda, l)f(z)) + \frac{\alpha}{p}z^{p+1}(\mathcal{L}_p^m(\lambda, l)f(z))'\right\} = \left\{p(z) + \frac{\alpha}{p}zp'(z)\right\}.$$

Since $f \in \sum_{k,p,n}(\alpha, m, \lambda, l, \rho)$, so $\left\{p(z) + \frac{\alpha}{p}zp'(z)\right\} \in P_k(\rho)$ for $z \in E$. This implies that

$$\Re\left\{p_i(z) + \frac{\alpha}{p}zp_i'(z)\right\} > \rho, \quad i = 1, 2.$$

Using Lemma 2.1, we see that $\Re\{p_i(z)\} > \rho_1$, where ρ_1 is given by (3.1). Consequently $p \in P_k(\rho_1)$ for $z \in E$, and the proof is complete. □

Now, we examine at the converse statement for Theorem 3.1.

Theorem 3.2. *Let* $f \in \sum_{k,p,n}(0, m, \lambda, l, \rho_1)$, *for* $z \in E$. *Then,* $f \in \sum_{k,p,n}(\alpha, m, \lambda, l, \rho)$ *for* $|z| < R(\alpha, p, n)$,
where

$$R(\alpha, p, n) = \left[\frac{p}{|\alpha|(p+n) + \sqrt{|\alpha|^2(p+n)^2 + p^2}}\right]^{\frac{1}{(p+n)}}. \tag{3.3}$$

Proof. Set

$$z^p(\mathcal{L}_p^m(\lambda, l)f(z)) = (p-\rho)h(z) + \rho, \qquad h \in P_k.$$

Now proceeding as in Theorem 3.1, we have

$$\left\{(1-\alpha)(z^p\mathcal{L}_p^m(\lambda, l)f(z)) + \frac{\alpha}{p}z^{p+1}(\mathcal{L}_p^m(\lambda, l)f(z))' - \rho\right\} = (p-\rho)\left\{h(z) + \frac{\alpha}{p}zh'(z)\right\}.$$

(3.4)

$$= (p-\rho)\left[(\frac{k}{4}+\frac{1}{2})\left\{h_1(z)+\frac{\alpha z h_1(z)}{p}\right\}-(\frac{k}{4}-\frac{1}{2})\left\{h_2(z)+\frac{\alpha z h_2(z)}{p}\right\}\right],$$

where we have used (1.5) and $h_1, h_2 \in P$, $z \in E$. Using the following well known estimates, see MacGregor [11],

$$\left|zh^{'}(z)\right| \leq \frac{2(p+n)\ r^{p+n}}{1-r^{2(p+n)}}\Re\{h(z)\}, \quad (|z|=r<1), \qquad i=1,2,$$

we have

$$\begin{aligned}\Re\left\{h_i(z)+\frac{\alpha}{p}zh_i'(z)\right\} &\geq \Re\left\{h_i(z)-\frac{|\alpha|}{p}\left|zh_i'(z)\right|\right\} \\ &\geq \Re h_i(z)\left\{1-\frac{2\left|\alpha\right|\ \ (p+n)r^{p+n}}{p(1-r^{2(p+n)})}\right\}.\end{aligned}$$

The right hand side of this inequality is positive if $r < R(\alpha, p, n)$, where $R(\alpha, p, n)$ is given by (3.3). Consequently it follows from (3.4) that $f \in \sum_{k,p,n}(\alpha, m, \lambda, l, \rho)$ for $|z| < R(\alpha, p, n)$.

Sharpness of this result follows by taking $h_i(z) = \frac{1+z^{p+n}}{1-z^{p+n}}$ in (3.4), $i = 1, 2$. □

Theorem 3.3.

$$\sum_{k,p,n}(\alpha_1, m, \lambda, l, \rho) \subset \sum_{k,p,n}(\alpha_2, m, \lambda, l, \rho) \text{ for } 0 \leq \alpha_2 < \alpha_1.$$

Proof. For $\alpha_2 = 0$, the proof is immediate. Let $\alpha_2 > 0$ and let $f \in \sum_{k,p,n}(\alpha_1, m, \lambda, l, \rho)$. Then, there exist two functions $H_1, H_2 \in P_k(\rho)$ such that, from Definition 1.1 and Theorem 3.1, we have

$$\left\{(1-\alpha)(z^p\mathcal{L}_p^m(\lambda,l)f(z))+\frac{\alpha}{p}z^{p+1}(\mathcal{L}_p^m(\lambda,l)f(z))'\right\}=H_1(z),$$

and

$$z^p(\mathcal{L}_p^m(\lambda,l)f(z))=H_2(z).$$

Hence,

(3.5)

$$\left\{(1-\alpha_2)(z^p\mathcal{L}_p^m(\lambda,l)f(z))+\frac{\alpha_2}{p}z^{p+1}(\mathcal{L}_p^m(\lambda,l)f(z))'\right\}=\frac{\alpha_2}{\alpha_1}H_1(z)+(1-\frac{\alpha_2}{\alpha_1})H_2(z).$$

Since the class $P_k(\rho)$ is a convex set, see Noor [12], it follows that the right hand side of (3.5) belongs to $P_k(\rho)$ and this proves the result. □

Theorem 3.4. *Let $f \in \sum_{k,p,n}(\alpha, m, \lambda, l, \rho)$, and let $\phi \in \sum_{p,n}$ satisfy the following inequality:*

$$\Re(z^p\phi(z)) > \frac{1}{2} \qquad (z \in E).$$

Then, $\phi * f \in \sum_{k,p,n}(\alpha, m, \lambda, l, \rho)$.

Proof. Let $F = \phi * F$. Then, we have

$$\begin{aligned} & \left\{(1-\alpha)(z^p\mathcal{L}_p^m(\lambda,l)F(z)) + \frac{\alpha}{p}z^{p+1}(\mathcal{L}_p^m(\lambda,l)F(z))'\right\} \\ = & \left\{(1-\alpha)(z^p\phi(z) * z^p(\mathcal{L}_p^m(\lambda,l)f(z)) + \frac{\alpha}{p}(z^p\phi(z) * z^{p+1}(\mathcal{L}_p^m(\lambda,l)f(z))')\right\} \\ = & (z^p\phi(z)) * G(z), \end{aligned}$$

where

$$G(z) = \left\{(1-\alpha)(z^p\mathcal{L}_p^m(\lambda,l)f(z)) + \frac{\alpha}{p}z^{p+1}(\mathcal{L}_p^m(\lambda,l)f(z))'\right\} \in P_k(\rho).$$

Therefore, we have

$$\begin{aligned} & (z^p\phi(z)) * G(z)(\frac{k}{4}+\frac{1}{2})\left\{(p-\rho)\left(z^p\phi(z) * g_1(z)\right)+\rho\right\} \\ & \quad -(\frac{k}{4}-\frac{1}{2})\left\{(p-\rho)\left(z^p\phi(z) * g_2(z)\right)+\rho\right\}, \\ & g_1,\ g_2 \in P. \end{aligned}$$

Since, $\Re\{(z^p\phi(z))\} > \frac{1}{2}$, $z \in E$, and so using Lemma 2.2, we conclude that $F = \phi * f \in \sum_{k,p,n}(\alpha, m, \lambda, l, \rho)$. □

Next, we study the interesting properties of the class $\mathcal{T}_{k,p,n}(\alpha, m, \lambda, l, \rho)$.

Theorem 3.5. *Let $f \in \mathcal{T}_{k,p,n}(\alpha, m, \lambda, l, \rho_2)$ and $g \in \mathcal{T}_{k,p,n}(\alpha, m, \lambda, l, \rho_3)$, and let $F = f * g$. Then, $F \in \mathcal{T}_{k,p,n}(\alpha, m, \lambda, l, \rho_4)$,*
where

$$\rho_4 = 1 - 4(1-\rho_2)(1-\rho_3)\left[1 - \frac{l}{\lambda\alpha}\int_0^1 \frac{u^{\frac{l}{\lambda\alpha}-1}}{1+u}du\right]. \tag{3.6}$$

Proof. Since, $f \in \mathcal{T}_{k,p,n}(\alpha, m, \lambda, l, \rho_2)$, it follows that

$$H(z) = \left\{(1-\alpha)(z^p\mathcal{L}_p^{m+1}(\lambda,l)f(z)) + \alpha(z^p\mathcal{L}_p^m(\lambda,l)f(z))\right\} \in P_k(\rho_2),$$

and so using identity (1.10) in the above equation, we have

$$(\mathcal{L}_p^{m+1}(\lambda,l)f(z) = \frac{l}{\lambda\alpha}z^{-p-\frac{l}{\lambda\alpha}}\int_0^z t^{\frac{l}{\lambda\alpha}-1}H(t)dt. \tag{3.7}$$

Similarly

$$(\mathcal{L}_p^{m+1}(\lambda,l)g(z) = \frac{l}{\lambda\alpha}z^{-p-\frac{l}{\lambda\alpha}}\int_0^z t^{\frac{l}{\lambda\alpha}-1}H^*(t)dt, \tag{3.8}$$

where $H^* \in P_k(\rho_3)$. Using (3.7) and (3.8), we have

$$(\mathcal{L}_p^{m+1}(\lambda,l)F(z) = \frac{l}{\lambda\alpha}z^{-p-\frac{l}{\lambda\alpha}}\int_0^z t^{\frac{l}{\lambda\alpha}-1}Q(t)dt, \tag{3.9}$$

where

$$Q(z) = \left(\frac{k}{4}+\frac{1}{2}\right) q_1(z) - \left(\frac{k}{4}-\frac{1}{2}\right) q_2(z) \tag{3.10}$$

$$= \frac{l}{\lambda\alpha} z^{-\frac{l}{\lambda\alpha}} \int_0^z t^{\frac{\lambda+p}{\alpha}-1}(H * H^*)dt.$$

Now

$$H(z) = \left(\frac{k}{4}+\frac{1}{2}\right) h_1(z) - \left(\frac{k}{4}-\frac{1}{2}\right) h_2(z),$$

$$H^*(z) = \left(\frac{k}{4}+\frac{1}{2}\right) h_1^*(z) - \left(\frac{k}{4}-\frac{1}{2}\right) h_2^*(z),$$

where $h_i \in P(\rho_2)$ and $h_i^* \in P(\rho_3)$, $i = 1, 2$.

Since

$$p_i^*(z) = \frac{h_i^*(z) - \rho_3}{2(1-\rho_3)} + \frac{1}{2} \in P(\frac{1}{2}), \quad i = 1, 2, \tag{3.11}$$

we obtain that $(h_i * p_i^*) \in P(\rho_3)$, by using Herglotz formula.

Thus,

$$(h_i * h_i^*) \in P(\rho_4), \tag{3.12}$$

with

$$\rho_4 = 1 - 2(1-\rho_2)(1-\rho_3).$$

Using $(3.9), (3.10), (3.11), (3.12)$ and Lemma 2.3, we have

$$\begin{aligned}
\Re q_i(z) &= \frac{l}{\lambda\alpha}\int_0^1 u^{\frac{l}{\lambda\alpha}-1} \Re\{(h_i * h_i^*)(uz)\}du \\
&\geq \frac{l}{\lambda\alpha}\int_0^1 u^{\frac{l}{\lambda\alpha}-1}\left(2\rho_4 - 1 + \frac{2(1-\rho_4)}{1+u\,|z|}\right)du \\
&\geq \frac{l}{\lambda\alpha}\int_0^1 u^{\frac{l}{\lambda\alpha}-1}\left(2\rho_4 - 1 + \frac{2(1-\rho_4)}{1+u}\right)du \\
&= 1 - 4(1-\rho_2)(1-\rho_3)\left[1 - \frac{l}{\lambda\alpha}\int_0^1 \frac{u^{\frac{l}{\lambda\alpha}-1}}{1+u}du\right].
\end{aligned}$$

From this we conclude that $F \in \mathcal{T}_{k,p,n}(\alpha, m, \lambda, l, \rho_4)$, where ρ_4 is given by (3.6).

We discuss the sharpness as follows:

We take

$$\begin{aligned}
H(z) &= \left(\frac{k}{4}+\frac{1}{2}\right)\frac{1+(1-2\rho_2)z}{1-z} - \left(\frac{k}{4}-\frac{1}{2}\right)\frac{1-(1-2\rho_2)z}{1+z}, \\
H^*(z) &= \left(\frac{k}{4}+\frac{1}{2}\right)\frac{1+(1-2\rho_3)z}{1-z} - \left(\frac{k}{4}-\frac{1}{2}\right)\frac{1-(1-2\rho_3)z}{1+z}.
\end{aligned}$$

Since,

$$\left(\frac{1+(1-2\rho_2)z}{1-z}\right)*\left(\frac{1+(1-2\rho_3)z}{1-z}\right) = 1-4(1-\rho_2)(1-\rho_3)+\frac{4(1-\rho_2)(1-\rho_3)}{1-z}.$$

It follows from (3.10), that

$$\begin{aligned} q_i(z) &= \frac{l}{\lambda\alpha}\int_0^1 u^{\frac{l}{\lambda\alpha}-1}\left\{1-4(1-\rho_2)(1-\rho_3)+\frac{4(1-\rho_2)(1-\rho_3)}{1-z}\right\}du \\ &\longrightarrow 1-4(1-\rho_2)(1-\rho_3)\left[1-\frac{l}{\lambda\alpha}\int_0^1 \frac{u^{\frac{l}{\lambda\alpha}-1}}{1+u}du\right] \qquad \text{as } z\longrightarrow -1. \end{aligned}$$

This completes the proof. □

Theorem 3.6. *Let $f(z)\in\sum_{p,n}$, we consider the integral operator J_c defined by*

$$\begin{aligned} J_cf(z) &= \frac{c}{z^{c+p}}\int_0^z t^{c+p-1}f(t)dt \\ &= \left(\frac{1}{z^p}+\sum_{t=n}^{\infty}\frac{c}{c+p+t}z^t\right)*f(z) \quad (c>0,\ z\in E^*). \end{aligned} \tag{3.13}$$

If

$$\left\{(1-\alpha)(z^p\mathcal{L}_p^{m+1}(\lambda,l)J_cf(z))+\alpha(z^p\mathcal{L}_p^{m+1}(\lambda,l)f(z))\right\}\in P_k(\rho), \tag{3.14}$$

then

$$(z^p\mathcal{L}_p^{m+1}(\lambda,l)J_cf(z))\in P_k(\beta), \quad z\in E,$$

where

$$\beta=\rho+(1-\rho)(2\gamma_1-1), \tag{3.15}$$

and

$$\gamma_1=\int_0^1\left(1+t^{\Re\frac{\alpha}{c}}\right)^{-1}dt.$$

Proof. First of all it follows from the Definition 3.13, that $J_cf(z)\in\sum_{p,n}$ and

$$z(\mathcal{L}_p^{m+1}(\lambda,l)J_cf(z))'=c(\mathcal{L}_p^{m+1}(\lambda,l)f(z))-(c+p)(\mathcal{L}_p^{m+1}(\lambda,l)J_cf(z)). \tag{3.16}$$

Let

$$(z^p\mathcal{L}_p^{m+1}(\lambda,l)J_cf(z))=h(z)=(\frac{k}{4}+\frac{1}{2})h_1(z)-(\frac{k}{4}-\frac{1}{2})h_2(z). \tag{3.17}$$

Then, the hypothesis (3.14) in conjection with (3.16) would yield

$$\left\{(1-\alpha)(z^p\mathcal{L}_p^{m+1}(\lambda,l)J_cf(z))+\alpha z^p(\mathcal{L}_p^{m+1}(\lambda,l)f(z))\right\}=\left\{h(z)+\frac{\alpha zh'(z)}{c}\right\}$$

$$\in P_k(\rho) \text{ for } z\in E.$$

Consequently

$$\left\{h_i(z) + \frac{\alpha z h_i'(z)}{c}\right\} \in P(\rho),\ i = 1, 2,\ 0 \leq \rho \leq p,\ \text{and} \quad z \in E.$$

Using Lemma 2.1, with $\lambda_1 = \frac{a}{c}$, we have $\Re\{h_i(z)\} > \beta$, where β is given by (3.15), and the proof is complete. □

Theorem 3.7. *Let $f \in \mathcal{T}_{k,p,n}(\alpha, m, \lambda, l, \rho)$, and let $\phi \in \sum_{p,n}$ satisfy the following inequality:*

$$\Re(z^p \phi(z)) > \frac{1}{2} \qquad (z \in E).$$

Then, $\phi * f \in \mathcal{T}_{k,p,n}(\alpha, m, \lambda, l, \rho)$.

Proof. Let $F = \phi * f$. Then, we have

$$\left\{(1-\alpha)(z^p \mathcal{L}_p^{m+1}(\lambda, l)F(z)) + \alpha z^p \mathcal{L}_p^m(\lambda, l)F(z)\right\} = z^p \phi(z) * G(z),$$

where

$$G(z) = \left\{(1-\alpha)(z^p \mathcal{L}_p^{m+1}(\lambda, l)f(z)) + \alpha(z^p \mathcal{L}_p^m(\lambda, l)f(z))\right\} \in P_k(\rho).$$

Therefore, we have

$$\begin{aligned} & z^p \phi(z) * G(z) \\ & = (\frac{k}{4} + \frac{1}{2})\{(p-\rho)(z^p \phi(z) * g_1(z)) + \rho\} - (\frac{k}{4} - \frac{1}{2})\{(p-\rho)(z^p \phi(z) * g_2(z)) + \rho\}, \\ & g_1, g_2 \in P. \end{aligned}$$

Since $\Re\{(z^p \phi(z))\} > \frac{1}{2}$, $z \in E$, and so using Lemma 2.2, we conclude that $F = \phi * f \in \mathcal{T}_{k,p,n}(\alpha, m, \lambda, l, \rho)$. □

Theorem 3.8. *For $0 \leq \alpha_2 < \alpha_1$,*

$$\mathcal{T}_{k,p,n}(\alpha_1, m, \lambda, l, \rho) \subset \mathcal{T}_{k,p,n}(\alpha_2, m, \lambda, l, \rho).$$

Proof. For $\alpha_2 = 0$, the proof is immediate. Let $\alpha_2 > 0$ and $f \in \mathcal{T}_{k,p,n}(\alpha_1, m, \lambda, l, \rho)$. Then,

$$\begin{aligned} & \left\{(1-\alpha_2)(z^p \mathcal{L}_p^{m+1}(\lambda, l)f(z)) + \alpha_2 z^p(\mathcal{L}_p^m(\lambda, l)f(z))\right\} \\ = & \frac{\alpha_2}{a_1}\left[\left(\frac{\alpha_1}{a_2} - 1\right)(z^p \mathcal{L}_p^{m+1}(\lambda, l)f(z)) + (1-a_1)(z^p \mathcal{L}_p^{m+1}(\lambda, l)f(z)) + a_1(z^p \mathcal{L}_p^m(\lambda, l)f(z))\right] \\ = & \left(1 - \frac{\alpha_2}{a_1}\right)H_1(z) + \frac{\alpha_2}{a_1}H_2(z), \qquad H_1, \quad H_2 \in P_k(\rho). \end{aligned}$$

Since $P_k(\rho)$ is a convex set, see Noor [12], we conclude that $f \in \mathcal{T}_{k,p,n}(\alpha_2, m, \lambda, l, \rho)$, for $z \in E$. □

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